DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

May 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) What is a normal family of analytic functions? What role does this concept play in the proof of the Riemann mapping theorem?

(b) State Picard's theoren on values omitted by entire functions. What role do analytic universal coverings play in its proof?

2. Show that if f is a nonvanishing analytic function on a simply-connected domain $G \subset \mathbb{C}$ then a single-valued analytic branch of $\log f$ can be defined on G.

3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2 + 2x + 5}\, dx$ via residues ($\lambda > 0$).

4. (a) What is the dual of $L^3(\mathbb{R})$?

(b) Show that the dual of ℓ^∞ ("bounded functions on the positive integers") is *not* ℓ^1 ("summable functions on the positive integers"), by exhibiting an element of the dual that is not in ℓ^1.

5. Let f be a non-negative integrable function on $\mathbb{R}$ (with Lebesgue measure), let μ be Lebesgue measure on $\mathbb{R}^2$, and show that

$$\mu\big(\{(x, y) : 0 \le y \le f(x)\}\big) = \mu\big(\{(x, y) : 0 < y < f(x)\}\big) = \int f(x)\, dx.$$

6. For some measures, $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, $L^r \supset L^s$, and for some measures, L^r can never contain L^s unless $r = s$. Find and explain examples of each phenomenon and/or necessary and/or sufficient conditions.

7. Let $\{\delta_n\}$ be a sequence of positive numbers, and $\{\phi_n\}$ an orthonormal set in an infinite-dimensional Hilbert space $\mathcal{H}$. Set

$$S = \{x = \sum_{n=1}^{\infty} a_n \phi_n \in \mathcal{H} :\mid a_n \mid \leq \delta_n\}.$$

Prove S is compact if and only if $\sum \delta_n^2 < \infty$. (In the case $\delta_n = \frac{1}{n}$, S is called the "Hilbert cube").

8. Find the maximum value of $\int_{-1}^{1} x^3 g(x) dx$, for measurable functions $g(x)$ satisfying

$$\int_{-1}^{1} g(x) dx = \int_{-1}^{1} x g(x) dx = \int_{-1}^{1} x^2 g(x) dx = 0,$$

and $\int_{-1}^{1} \mid g(x) \mid^2 dx = 1$.

9. Evaluate the derivative and the second derivative of the Heaviside function H on $\mathbb{R}$, in the sense of distributions; the Heaviside function is:

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$