DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 7, 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) Let S be the class of all complex-valued measurable simple functions s on a set X with measure μ such that $\mu\{x \mid s(x) \neq 0\} < \infty$. If $1 \leq p < \infty$ show that S is dense in $L^P(\mu)$.

(b) Give an example of a set X, a measure μ, and a sequence of functions $\{f_n\}_{n=1}^{\infty}$ which converges to 0 in $L^P(\mu)$ but does not converge to 0 pointwise a.e.

2. Consider a Hilbert Space H, and a subspace M.

(a) Give an example of H and M such that M is not closed.

(b) Prove that $M^{\perp}$ is a *closed* subspace of H.

(c) Prove that $\overline{M} = M^{\perp\perp}$.

3. (a) Give a necessary and sufficient condition for a set $E \subset R$ to have Lebesgue measure 0.

(b) State the definition of absolutely continuous measures, and singular measures.

(c) Find a measure μ, singular with respect to Lebesgue measure, such that $\mu(I) > 0$ for every non-empty interval I.

(d) For a measure μ on R, satisfying

$$\left| \int e^{2\pi i n x} \, d\mu(x) \right| \leq C \, |n|^{-2}, \qquad n \neq 0,$$

prove that it is absolutely continuous with respect to the Lebesgue measure.

4. (a) Is the following a Banach Space (with respect to a suitable norm)?

$$B \; = \; \{f : R \to R \quad \text{s.t.} \quad f \text{ continuous, and } \lim_{|x|\to\infty} f(x) = 0\}\,.$$

Justify your answer.

(b) Suppose f is continuous, and such that

$$\sup|f \cdot g| \;\leq\; C \sup|g|, \qquad \text{for all } f \in B\,.$$

Prove that

$$|f| \;\leq\; C\,.$$

5. Suppose that f is analytic in $|z| < R_0$ and that $|a| < R < R_0$. Evaluate

$$\int_\gamma \frac{(R^2 - |a|^2)f(z)}{(z-a)(R^2 - z\bar{a})}\,dz$$

where γ is the circle $|z| = R$ transversed counterclockwise. Hence prove that if $0 < r < R$

$$f(re^{i\theta}) \;=\; \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2}$$

(Poisson's formula).

6. (a) Show that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right)$ then f is analytic in the whole plane minus the points $0, 1, 2, \ldots$.

(b) Find $f^{(k)}(z)$. (Justify.)