DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

May 1, 2000

No aids.
Do all questions.
The value of each is indicated.
For each part provide a complete justification of your claims.
Total = 120. $120 \times \frac{5}{6} = 100$.

1. [5 × 4 = 20 marks]

Prove or give counterexample:

(a) Any open set $U \subset (0,1)$ which includes all rationals in the unit interval has full Lebesgue measure: $m(U) = 1$.

(b) The Banach space dual of $L^\infty([0,1])$ is $L^1([0,1])$.

(c) If $f_n \to f$ in $L^1(\mathbf{R})$, then f_n converges pointwise a.e.

(d) A non-decreasing function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is differentiable a.e.

2. [10 × 2 = 20 marks]

Let $\psi(x) \geq 0$ be a smooth, compactly supported function satisfying $\int_{-\infty}^{\infty} \psi(x)\,dx = 1$. Given any (real-valued function) $f \in L^1(\mathbf{R})$ define its convolution with ψ by

$$g(x) = \int_{\mathbf{R}} f(y)\psi(x-y)dy.$$

(a) Prove $g(x)$ is differentiable everywhere.

(b) Prove $\|g\|_1 \leq \|f\|_1$ for real $f \in L^1(\mathbf{R})$ and discuss the cases of equality.

3. [10 × 2 = 20 marks]

(a) Given a closed subspace $K \subset H$ of a Hilbert space H, prove that each $h \in H$ can be uniquely expressed as $h = k + j$, where $k \in K$ and j is orthogonal to K.

(b) Among $f \in L^2[0,1]$ satisfying $\|f\|_2 = 1$, which function(s) maximize $\int_0^1 xf(x)\,dx$?

4. (a) **[5 marks]**

Compute the Fourier transform of the Gaussian $g(x) = e^{-x^2/2}$.

(b) **[5 marks]**

Compute the Fourier transform of the error function

$$\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

(c) **[10 marks]**

Construct an elementary function whose Fourier transform is continuous but fails to be differentiable.

5. (a) **[6 marks]**

Evaluate $\int_\gamma \frac{z\,dz}{z^4-1}$ where γ is the circle $|z - ia| = a$ traversed once counterclockwise and $a > 1$.

(b) **[6 marks]**

Evaluate $\int_\gamma \frac{ze^z}{(z-b)^3} dz$ where γ is the unit circle traversed once counterclockwise and $|b| \neq 1$.

(c) **[8 marks]**

Let U be the open unit disc, i.e. $U = \{z \in \mathbb{C} \mid |z| < 1\}$. Is $U \backslash \{0\}$ biholomorphic to $\mathbb{C} \backslash \{0\}$? Explain.

6. **[20 marks]**

A family $\mathcal{F}$ of holomorphic functions on a domain Ω is said to be normal if any sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω. Suppose that the family $\mathcal{F}$ fails to be normal. Show that there exists a point $z_0 \in \Omega$ such that $\mathcal{F}$ is not normal in any neighbourhood of Ω.

Hint: a compactness argument.