DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 5, 2000

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) In each of the following, identify the completion of the set C of continuous real-valued functions on $[0,1]$ with respect to the given notion of convergence:

(i) $\{f_n\}_{n=1}^{\infty} \subset C$, $f \in C$, $f_n \to f$ if $\lim\limits_{n\to\infty} \int\limits_0^1 |f_n(x) - f(x)|^2 dx = 0$.

(ii) $\{f_n\}_{n=1}^{\infty} \subset C$, $f \in C$, $f_n \to f$ if $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)g(x)dx = \int\limits_0^1 f(x)g(x)dx$ for all $g \in C$.

(b) Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions on $[0,1]$ converging pointwise to 0. Determine which of the following conditions are sufficient to ensure that $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)dx = 0$. Prove your answer.

(i) All functions f_n are positive and absolutely continuous.

(ii) For every n, f_n is integrable and for every positive x $|f_n(x)| < \frac{1}{x}$.

(iii) For every n, f_n is integrable and for every positive x $|f_n(x)| < \frac{1}{\sqrt{x}}$.

2. Let S be the set of all complex-valued functions f on $\mathbb{R}$ such that $f \in C^{\infty}(\mathbb{R})$ and there are numbers $A_{mn} = A_{mn}(f)$ such that

$$\left| x^n \frac{d^m f}{dx^m} \right| \leq A_{mn}, \qquad x \in \mathbb{R},\ m,n = 0,1,2,\ldots$$

Prove that the Fourier transform maps S onto S. Find examples of members of S.

3. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi \geq 0$ and $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. For $\lambda > 0$ define $\varphi_\lambda(x) = \lambda^{-1}\varphi(x/\lambda)$ for $x \in \mathbb{R}$. If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, prove that $\lim\limits_{\lambda \to 0} \|f * \varphi_\lambda - f\|_p = 0$.

4. (a) Does there exist a Banach space X and a linear functional A: $X \to \mathbb{R}$ such that $ker\, A$ is **not** closed in X? Justify your answer.

(b) State the following carefully and precisely:

(i) the Radon-Nikodym theorem

(ii) Fubini's theorem

5. Suppose $\alpha \in \mathbb{C}$, $|\alpha| \neq 1$. Compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

by integrating $(z-\alpha)^{-1}(z - \frac{1}{\alpha})^{-1}$ around the unit circle.

6. Show that the set of biholomorphic maps from the upper half plane onto itself consists precisely of the set of linear fractional transformations of the form $\frac{az+b}{az+d}$ where $a, b, c, d \in \mathbb{R}$ (after cancellation of a common factor) and $ad - bc > 0$.