DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Tuesday, September 4, 2001, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. State the following carefully and precisely:

(a) The Cauchy integral formula for the 10th derivative of an analytic function.

(b) The Poisson integral formula for a real harmonic function.

(c) The little Picard Theorem.

(d) Rouché's Theorem.

(e) The spectral theorem for a self-adjoint compact operator.

2. What is the general form of a rational function (the quotient of two complex polynomials) which is of real value on the unit circle? In particular, how are the zeros and the poles located?

3. Prove or disprove (i.e. give a counterexample) for each of the following.

(a) Each measurable set $A \subset [0,1]$ has the same Lebesgue measure as the topological closure of A.

(b) The orthogonal complement of any linear subspace (closed or not) of a Hilbert space H is a closed linear subspace of H.

(c) Let $\{f_n\}$ be a sequence in $L^1([0,1])$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)f(x)dx = 0$ for every f in $L^1([0,1])$. Then $\lim_{n\to\infty} \int_0^1 |f_n(x)|dx = 0$.

4. Let μ denote the Lebesgue measure on [0,1]. Prove that every Cauchy sequence in $L^p(\mu)$ with $1 \le p < \infty$ has a subsequence that converges pointwise almost everywhere on $[0,1]$.

5. Let K denote the set of all real continuous functions on $[0, 1]$ that satisfy:

(i) $|f(x)| \leq 1$ for each $f \in K$ and all $x \in [0, 1]$.

(ii) $|f(x) - f(y)| \leq |x - y|$ for all x, y in $[0, 1]$ and each f in K.

Prove that K is sequentially compact in $C([0, 1])$. Here $C[0, 1]$ is the vector space of real continuous functions on $[0, 1]$ topologized by the sup norm.

6. (a) Find the Fourier transform of $xe^{-\frac{x^2}{2}}$.

(b) What is the Fourier transform of $x^n e^{-\frac{x^2}{2}}$.

(c) Interpret your results in (a) and (b) in terms of the eigenvalues of $\mathcal{F}$: $L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ with $\mathcal{F}(f) =$ the Fourier transform of f.

(d) Give a complete set of eigenvalues for $\mathcal{F}$.