DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Monday, May 6, 2002, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. Prove or disprove (i.e., find a counterexample) each of the following statements:

(a) if $f \in L^1([0,1])$ then $\lim_{n\to\infty} \int_0^1 f(x) \cos n\pi x dx = 0$.

(b) $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$ for $1 \le p < q$.

(c) Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ that converges pointwise to zero. Then $\lim_{n\to\infty} \int_0^1 f_n dx = 0$ whenever $x|f_n(x)| \le \sqrt{x}$ for all $x > 0$.

2. (a) State the Riesz representation theorem for $L^p(\mu)$ spaces with $1 \le p < \infty$. Here μ denotes a positive measure on a measure space X.

(b) Let f be a measurable function such that the product fg is in $L^1(\mu)$ for each $g \in L^q(\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f \in L^p(\mu)$.

3. Let X denote a Banach space and let x_0 be a non zero element in X. Show that there exists a bounded linear functional f such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

Also show that for any distinct points x and y in X there exists a bounded linear functional f such that $f(x) \neq f(y)$.

4. Let H denote a real Hilbert space and let M be a linear subspace.

(a) Give an example of H and M in which M is not closed in H.

(b) Suppose that M is a closed subspace of H and suppose that x_0 is a point in H not in M. Prove that

$$\text{Minimum}\{\|x - x_0\| : \ x \in M\} = \text{Maximum}\{\langle x, x_0 \rangle : \ x \in M^{\perp}, \ \|x\| = 1\}.$$

Here, $\langle \ , \ \rangle$ denotes the inner product on H, and $M^{\perp}$ denotes the orthogonal complement of M.

5. (a) Use the theory of residues to calculate the integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$$

(b) What is the image of the unit disk in the w-plane under the mapping

$$w = z + \frac{1}{z}.$$

6. Let f be an analytic function in the punctured disk $\Delta = \{z :\ 0 < |z - z_0| < \Omega\}$ that has an essential singularity at z_0. Prove that $f(\Delta)$ is dense in $\mathbb{C}$.