DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Tuesday, September 3, 2002, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. Prove or find a counterexample for each of the following statements. Here, all integrals are defined by the Lebesgue measure m on the real line.

(a) If A is a measurable set then there exists a sequence of open sets $U_n \subseteq A$ such that $m(A) = \lim_{n\to\infty} m(U_n)$.

(b) If $\int_{-1}^{1} x^n f(x)dx = 0$ for each $n = 0, 1, \dots$ where f is a bounded and measurable function on $[-1, 1]$ then $f = 0$ a.e.

(c) $f(b) - f(a) = \int_a^b f'(x)dx$ for any continuous and increasing function f.

2. (a) Prove that $f \in L^q(\mathbb{R})$ for any q such that $p_1 \leq q \leq p_2$ whenever $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$. Here, $0 < p_1 < p_2$ and the L^p spaces are relative to the Lebesgue measure on $\mathbb{R}$.

(b) Show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ whenever $p \neq q$.

Which of the above statements becomes false if the real line is replaced by a finite interval $[a, b]$. Explain.

3. Let $C[0, 1]$ denote the space of continuous functions on $[0, 1]$ equipped with the sup-norm, and let K be a continuous function on $[0, 1] \times [0, 1]$. Suppose that $L\colon C[0, 1] \to C[0, 1]$ is defined by $Lf = g$ if and only if $g(y) = \int_0^1 K(x, y)f(x)dx$ for all $y \in [0, 1]$. Prove that $\{Lf_n\}$ contains a convergent subsequence in $C[0, 1]$ for any bounded sequence $\{f_n\}$ in $C[0, 1]$.

4. Find the maximum value of $\int_{-1}^{1} x^5 f(x)dx$ among all Lebesgue measurable functions f on $[-1, 1]$ such that $\int_{-1}^{1} f^2 dx = 1$, and $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} xf(x)dx = \int_{-1}^{1} x^2 f(x)dx = 0$.

5. Use the theory of residues to evaluate

$$\int_0^\infty \frac{\cos\, x}{(x^2+b^2)^2}\, dx \qquad \text{where } b > 0.$$

6. Let h_n be a sequence of harmonic functions which converge uniformly on compact subsets of the open set $U \subset \mathbb{C}$ to a function h. Prove that $\dfrac{\partial h_n}{\partial x}$ converges to $\dfrac{\partial h}{\partial x}$ uniformly on compact subsets of U.