DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Monday, May 5, 2003, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. Let $(E, \mathcal{B}, \mu)$ be a measure space. The measure μ is σ-finite if there exists $\{E_n\} \subset \mathcal{B}$ such that

$$E = \cup_{n=1}^{\infty} E_n \qquad \text{and} \qquad \mu(E_n) < \infty \, \forall n.$$

Prove that μ is σ-finite if and only if there exists $f > 0$, $f \in L^1(\mu)$.

2. Let H be a Hilbert space with orthonormal basis $\{\phi_k\}$. $\mathcal{L}(H, H)$ is the Banach space of bounded linear operators from H to H with the norm on $\mathcal{L}(H, H)$ being induced by the norm on H:

$$\|A\|_{\mathcal{L}(H,H)} = \sup_{x \neq 0} \frac{\|Ax\|_H}{\|x\|_H}$$

$A \in \mathcal{L}(H, H)$ is a *compact* operator if

$$\overline{AU}$$

is compact whenever U is a bounded subset of H. That is, the closure of the image of a bounded set is compact.

a) Let

$$\psi_n \in [\text{span}\{\phi_1, \ldots, \phi_n\}]^{\perp} \qquad \text{and} \qquad \|\psi_n\| = 1.$$

Prove that the sequence ψ_n converges weakly to 0.

b) Assume that the sequence $\{x_n\} \subset H$ converges weakly to x. Assume A is a compact operator. Prove that the sequence $\{Ax_n\}$ converges strongly to Ax. (i.e., $\|Ax_n - Ax\| \to 0$).

c) Assume A is a compact operator. Construct a sequence of finite-rank operators $\{A_n\}$ such that

$$\lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(H,H)} = 0.$$

3. Let $f : [0,1] \to [0,1]$ be continuously differentiable and satisfy $f(0) = 0$ and $f(1) = 1$.

a) Let
$$A_n = \{x \in [0,1] \mid |f'(x)| < 1/n\} \quad \text{and} \quad B_n = f(A_n).$$
Prove that $\mu(B_n) \leq 1/n$ where μ is Lebesgue measure.

b) A point x_0 is a critical point of f if $f'(x_0) = 0$. The image of a critical point, $f(x_0)$ is called a critical value. Prove that the set of critical values of f has Lebesgue measure zero.

c) Prove there exists at least one horizontal line $y = y_0 \in [0,1]$ which is nowhere tangent to the graph of f in $\mathbf{R}^2$. (Recall that the graph of f is the set of points $\{(x, f(x))\}$.)

4. Let X be a complex Banach space. Let I denote the identity operator. We say that $B \in \mathcal{L}(X,X)$ is invertible if B is injective, onto, and $B^{-1} \in \mathcal{L}(X,X)$.

a) Let $S, T \in \mathcal{L}(X,X)$. Prove that $I - ST$ is invertible if and only if $I - TS$ is invertible. (*Hint: do some formal manipulations using geometric series to try and write one inverse in terms of the other.*)

b) Let $S, T \in \mathcal{L}(X,X)$. Prove that
$$\text{spectrum}(ST) - \{0\} = \text{spectrum}(TS) - \{0\}.$$

c) Let $S, T \in \mathcal{L}(X,X)$. Prove that $ST - TS \neq I$. (*Hint: assume $ST - TS = I$ and then prove the spectrum of ST must be unbounded and then ...*)

5. Let $f(z) = \dfrac{g(z)}{h(z)}$ where g, h are analytic in a neighbourhood of z_0, $g(z_0) \neq 0$ and $h(z_0) = h'(z_0) = 0$. Show that the residue of f at z_0 is given by
$$\frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{\left(h''(z_0)\right)^2}.$$

6. Let $\mathcal{F}$ denote the family of circles and lines in the plane.

a) Prove that any linear fractional transformation maps members of $\mathcal{F}$ to members of $\mathcal{F}$.

b) What is the image of the circle with center 0 and radius 2 under the mapping $z \to \dfrac{z-i}{z+i}$?