DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Tuesday, September 2, 2003, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) Let E be a normed real vector space, $x_0 \in E$. Prove there exists a linear functional ϕ on E so that $\phi(\alpha x_0) = -3\alpha$ for all $\alpha \in \mathbb{R}$.

(b) Assume E has an inner product. Use the inner product to write your ϕ in an explicit manner.

2. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx = 0.$$

3. Let X be a Banach space and $A : X \to X$ a bounded operator. Recall that $\mathcal{L}(X, X)$ is the Banach space of bounded linear operators from X to X with the norm induced by the norm on X.

a) Fix $t \in \mathbb{R}$. Construct

$$e^{tA} \in \mathcal{L}(X, X).$$

(That is, define an operator B that is the most sensible definition of e^{tA} that you can think of and prove that $B \in \mathcal{L}(X, X)$.)

b) Given $x_0 \in X$ we define a path $x(t) \in X$ for $t \in \mathbb{R}$ by

$$x(t) = e^{tA} x_0.$$

Prove that

$$\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$

exists in X and call this limit "dx/dt at time t". Prove that at each time t

$$\frac{dx}{dt}(t) = Ae^{tA}x_0 = Ax(t).$$

4. Let H be a Hilbert space and $A : H \to H$ be a bounded linear operator. The point spectrum of A is:

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid Ax = \lambda x, \quad \text{for some } x \in H,\ x \neq 0\}$$

Prove or disprove:

$$\sup\{|\lambda| : \ \lambda \in \sigma(A)\} = \|A\|.$$

5. (a) State Schwarz's Lemma.

(b) Prove that every 1 - 1 analytic mapping from $\Delta := \{z \mid |z| < 1\}$ onto Δ is of the form

$$f(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) \quad \text{for some } \alpha \in \Delta.$$

6. (a) Define normal family (of analytic functions) and state a general theorem which gives a criterion for a family of analytic functions to be normal.

b) Consider $\left\{f \mid f = \sum_{n=0}^{\infty} a_n z^n \text{ with } |a_n| \leq n \text{ for } n = 1, 2, \dots\right\}$.

Using (a) above (or otherwise) show that this is a normal family of analytic functions.