DEPARTMENT OF MATHEMATICS

University of Toronto

Analysis Exam (3 hours)

Friday, May 7, 2004, 1–4 p.m.

All problems have equal weight.

1. Prove or find a counterexample for each of the following statements. Here μ denotes the Lebesgue measure on the real line.

   (a)Let $\{\phi_n\}$ denote a sequence of functions on $[0,1]$ such that

$$\int_0^1 |\phi_n|\,d\mu \le C$$

   for some constant C, and

$$\lim_{n\to\infty}\int_E \phi_n\,d\mu = 0 \text{ for each measurable subset E of [0,1].}$$

   Then, $\lim\limits_{n\to\infty}\int_0^1 f\phi_n\,d\mu = 0$ for each bounded and measurable funtion f on $[0,1]$.

   (b) If $\{f_n\} \subset L^2([0,1])$ and $\lim\limits_{n\to\infty}\int_0^1 f_n^2\,d\mu = 0$, then $\lim\limits_{n\to\infty} f_n(x) = 0$ for almost all $x \in [0,1]$.

2. Let μ denote a finite Borel measure on R. Evaluate the following limits:

$$\lim_{t\to\infty}\int_R f(tx)\,d\mu(x)$$

$$\lim_{t\to 0}\int_R f(tx)\,d\mu(x),$$

   where f denotes a continuous function on R with compact support. Justify your answers.

3. Let H denote a real Hilbert space with an inner product $< x, y >$. Consider the function $Q(x, y) =< x, Ay >$ on $H \times H$ for some linear transformation A on H.

   (a) Show that Q is continuous if and only if A is continuous.

   (b) If Q is continuous show that

$$\sup \frac{|Q(x,y)|}{||x|| ||y||} = \sup \frac{||A(x)||}{||x||}, \; x \neq 0, y \neq 0.$$

4. Let M denote a closed linear subspace of a Banach space X and let x be any point of X not in M.

   (a) Show that $\inf\{||x - y|| : y \in M\} = \delta > 0$

   (b) Show that there exists a bounded linear functional f on X such that $f(x) = ||x||$ and $f(y) = 0$ for all $y \in M$.

5. (a) Define normal family of analytic functions.

   (b) Let $\mathcal{F} = \{f \mid f$ is holomorphic in $|z| < 1$ and $|f^{(n)}(0)| \leq n!$ for $n = 0, 1, 2, \ldots\}$. Prove that $\mathcal{F}$ is a normal family.

6. (a) State Schwarz's Lemma.

   (b) Let f be a 1–1 analytic mapping from the unit disc $\Delta = \{z \mid |z| < 1\}$ onto itself such that $f(0) = 0$. Prove that f is of the form $f(z) = e^{i\theta} z$ for some θ.