ANALYSIS PhD Comprehensive Exam
September 7, 2004, 2–5 p.m.

All problems have equal weight.

Notations. Here m denotes the Lebesgue measure on the real line. For any subset of real numbers A and any real number x, $A + x$ designates the translate $\{a + x : a \in A\}$. The symmetric difference of sets $A \cup B - A \cap B$ will be denoted by $A \Delta B$. .

1. Show that
$$\lim_{n\to\infty} m((A + \frac{1}{n})\Delta A) = 0$$
for each Lebesgue measurable set A of finite measure. Also show by an example that the above may fail for sets of infinite measure.

2. Let X be a measurable subset of real numbers. Recall that $L^p(X) = \{f : \int_X |f|^p \, dm < \infty\}$ for each real $p \in [1, \infty)$.

   (a) Show that $L^p(X) \subseteq L^r(X)$ whenever $m(X) < \infty$ and $1 \le r < p$.

   (b) Assume that $m(X) = \infty$. Show that $L^p(X)$ is not a subspace of $L^r(X)$ for $r \neq p$.

3. Let $K(x, y)$ denote a continuous function on the square $[0, 1] \times [0, 1]$. For each f in $L^2([0, 1])$ let Tf be the function on the interval $[0, 1]$ defined by
$$Tf(x) = \int_0^1 K(x, y) f(y) \, dy$$

   (a) Show that Tf is continuous for each f.

   (b) If f_n denotes a sequence of functions in the unit ball in $L^2([0, 1])$ show that the sequence of functions Tf_n contains a uniformly convergent subsequence.

4. Suppose that g is a real measurable function on the interval $[0, 1]$ such that
$$\int_0^1 g(x) f(x) \, dx < \infty$$

for each real measurable function f such that $\int_0^1 f^2(x)\,dx < \infty$. Show that $\int_0^1 g^2(x)\,dx < \infty$.

5. (a) Define normal family of analytic functions.

   (b) Let Ω be a connected open set in $\mathbb{C}$. Let $\{f_n\}$ be a sequence of polynomials in z, each of degree $\leq D$. Suppose that the sequence $\{f_n\}$ converges uniformly on compact subsets of Ω to a function of f. Prove that f is a holomorphic polynomial of degree $\leq D$.

6. (a) State Schwarz's Lemma.

   (b) Let f be an analytic mapping from the unit disc $\Delta = \{z \mid |z| < 1\}$ to itself satisfying $f(a) = b$ for some points $a, b \in \Delta$. Prove that

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

   Hint: Consider $g := \phi_b \circ f \circ \phi_{-a}$ where

$$\phi_\alpha := \frac{z - \alpha}{1 - \overline{\alpha} z}.$$