

University of Toronto  
**Department of Mathematics**  
**Analysis Examination**

Monday, May 2, 2005, 1–4 p.m.  
Duration 3 hours

No aids allowed.

All questions are equal in value.

In case a problem contains an assertion the problem is to prove that assertion.

1. (a) A Hilbert space  $\mathcal{H}$  is separable if and only if it has a countable orthonormal basis.  
(b) Suppose  $\mathcal{H}$  is any separable infinite-dimensional Hilbert space. Then  $\mathcal{H}$  has a family of closed subspaces  $\{E_t : t \in [0, 1]\}$  such that  $E_s$  is a strict subspace of  $E_t$  for all  $0 \leq s < t \leq 1$ .
2. In this problem  $(X, \mu)$  is any finite measure space.  
(a) Suppose  $f_n \in L_1(\mu)$  and  $\|f_n\|_1 \leq 1$  for all  $n$ . Prove or disprove:  $f_n/n \rightarrow 0$  almost everywhere as  $n \rightarrow \infty$ .  
(b) Suppose  $f_n$  are measurable functions on  $X$  with values in  $(-\infty, \infty)$ . (The  $f_n$  are not assumed to be integrable.) Then there are constants  $c_n > 0$  such that  $c_n f_n \rightarrow 0$  almost everywhere.
3. In this problem  $X$  is a Banach space and  $P : X \rightarrow X$  is a (not necessarily continuous) linear map such that  $P^2 = P$ . Let  $R = P(X)$  denote the range of  $P$  and  $N$  the kernel of  $P$ .  
(a)  $X$  is the direct sum of  $P$  and  $N$ , that is  $X = P + N$  and  $P \cap N = \{0\}$ .  
(b)  $P$  is continuous if and only if  $R$  and  $N$  are closed.
4. In this problem  $g$  is a continuous  $2\pi$ -periodic function on  $\mathbb{R}$ . For  $n \in \mathbb{Z}$  the  $n$ -th Fourier co-efficient of  $g$  is

$$\widehat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

and the convolution  $g * g$  is defined by

$$g * g(x) = \frac{1}{2\pi} \int_0^{2\pi} g(x - y)g(y)dy.$$

State precisely any facts about the Fourier transform which you use in your arguments.

- (a)  $g \in C^\infty(\mathbb{R})$  if and only if for each  $k > 0$  there is a constant  $C_k$  such that  $|\widehat{g}(n)| \leq C_k |n|^{-k}$ .

- (b) If  $g * g \in C^\infty(\mathbb{R})$  then  $g \in C^\infty(\mathbb{R})$ .
5. Let  $f$  denote a holomorphic mapping of the unit disk to itself which is not the identity mapping. Show that  $f$  can have at most one fixed point.
6. Let  $f$  denote an entire function (which is not identically zero). Let  $n$  be a natural number. Prove that the following are equivalent:
- (a) There exists an entire function  $g$  such that  $f = g^n$ .
  - (b) Every zero of  $f$  has order divisible by  $n$ .