University of Toronto Department of Mathematics

Analysis Examination

Tuesday, September 5, 2006, 1-4:00 p.m. Duration 3 hours

Instructions: Do all questions. You may use a previous part of a question to solve a subsequent part even if you have not done the previous part. Do not be concerned if you can't do all the problems. Good luck

- 1. [25 marks] Quickies
 - (a) Prove that if T is a linear transformation of a Hilbert space \mathcal{H} into itself such that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$ then T is bounded.
 - (b) Suppose C is a convex subset with non-empty interior in a normed vector space X. Show that the interior of C is dense in C. (Hint: draw a picture.)
 - (c) Suppose $\{a_i\}$ is a sequence in l_2 and a_i is never zero. Show that there is a $\{b_i\} \in l_1$ such that $\{\frac{b_i}{a_i}\}$ is not in l_2 . (Hint: restate this as a question about a linear operator between Banach spaces.)
- 2. [15 marks]
 - (a) Suppose $f,g\in L^2(\mathbb{R})$. Show that the convolution $\,f\ast g\,$ is an everywhere defined continuous function.
 - (b) Suppose $f \in L^1(\mathbb{R})$ and is not equal a.e. to the zero function. Show that f * f is not identically zero. (Hint: use properties of the Fourier transform.)
 - (c) Deduce from (b) that if $A, B \subset \mathbb{R}$ are subsets of positive measure then the set A+B contains a non-empty open interval. (Hint: it is easy to reduce to the case when A = B and A has finite measure.)
- 3. [15 marks] Suppose $1 \le p < \infty$ $f_n, f \in L_p[0,1], f_n \to f$ almost everywhere and $\|f_n\|_p \to \|f_p\|$. Prove or disprove: $\|f_n f\|_p \to 0$
- 4. [15 marks] Let m denote Lebesgue measure on \mathbb{R}^2 and suppose that $A\subset\mathbb{R}^2$ with $m(A)<\infty$.
 - (a) Prove that $m(A\Delta(A+x)) \to 0$ as $x \to 0$ in \mathbb{R}^2 . Here Δ denotes the symmetric difference of set. (Hint: this is clear if $A = I \times J$ where I and J are bounded intervals in \mathbb{R} . Proceed by an approximation argument.)

- (b) Suppose m(A) > 0. Use part (a) to show that there is an $\epsilon > 0$ such that for each $\delta < \epsilon$, A contains the vertices of some square of side δ . (Hint: show that the intersection of four suitable translates of A has positive measure, and hence is non-empty.)
- 5. [15 marks] State each of the following carefully and precisely.
 - (a) The definition of an essential singularity as an isolated singularity of an analytic function.
 - (b) Liouville's Theorem.
 - (c) Laurent's Theorem about the Laurent series for a function analytic on an open annulus.
 - (d) Rouché's Theorem about the zeros of two analytic functions.
 - (e) The Little Picard Theorem.
- 6. [15 marks] Evaluate the following integrals by means of the residues

(a)
$$\int_{-\pi}^{\pi} \frac{d\theta}{5+3\cos\theta}$$
 (b) $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3} dx$.