University of Toronto

Department of Mathematics

Analysis Examination

Tuesday, September 4, 2007, 1–4 p.m.

Duration 3 hours

1. Find the real values of a, b which minimize $\int_1^\infty |x^{-2} - ax^{-3} - bx^{-4}|^2\, dx$. Hint: Work in an appropriate Hilbert space.

2. Suppose F is a closed subset of $(0,1)$, $G = (0,1)\backslash F$ and let

$$d(x,F) = \inf_{y \in F} |x - y|$$

denote the distance from x to F. Let

$$M(x) = \int_0^1 \frac{d(y,F)}{|x-y|^2} dy.$$

   (a) Show that $M(x) = \infty$ for all $x \in G$.

   (b) Show that $M(x) < \infty$ for almost all $x \in F$ by showing that $\int_F M(x)dx < 2\mu(G)$, where μ denotes Lebesgue measure on $(0,1)$. Hint: The integral defining $M(x)$ may be restricted to elements y in G. Reverse the order of integration and then bound the inner integral by observing that for each $y \in G$ one has $F \subset \{x : |x-y| \geq d(y,F)\}$.

3. Suppose X is a Banach space. A projection on X is a linear map $P : X \to X$ such that $P^2 = P$.

   (a) Show that $I - P$ is also a projection whose kernel is the range of P. (I denotes the identity map on X).

   (b) Show that the range and the kernel of P span X.

   (c) Show that P is continuous if and only if the range and kernel are closed subspaces of X. (One direction is easy and for the other you can use a non-trivial theorem about the continuity of linear maps on Banach spaces.)

4. In this problem L_p denotes the L_p-space of $[0, 2\pi]$ endowed with Lebesgue measure. For $f \in L_1$ $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ denotes the nth Fourier coefficient of f. Suppose $f \in L_1$, $\hat{f}(0) = 0$ and let $F(t) = \int_0^t f(s)ds$.

   (a) Show that $\hat{F}(n) = \frac{1}{in}\hat{f}(n)$.

   (b) Supppose that $f \in L_2$. Show that $\sum_{n\in\mathbb{Z}} |\hat{F}(n)| < \infty$. Hint: Use part (a) and the Cauchy-Schwartz inequality.

5. Does there exist a function $f(z)$ holomorphic in $\mathbb{C} \setminus \{0\}$, such that

$$|f(z)| \geq \frac{1}{\sqrt{|z|}} ,$$

for all $z \in \mathbb{C} \setminus \{0\}$? Why?

6. Use residues to show that

$$\int_0^1 \frac{dx}{\sqrt[3]{x^2 - x^3}} = \frac{2\pi}{\sqrt{3}} .$$