Mathematics Department
University of Toronto
Analysis Examination
September 2009
Duration 3 hours
No aids allowed

1. Let f be an L^p function on $\mathbb{R}$. If $p > 4/3$, prove that

$$\lim_{t \to 0^+} \int_0^t x^{-1/4} f(x) dx = 0.$$

2. Let A, B be measurable subsets of $[0,1]$ with $m(A) = m(B) = 1/4$. For any real number t, let B_t denote the translation of B by t. In other words, $B_t = \{b+t\}_{b \in B}$.
   Prove that there exists $t \in \mathbb{R}$ so that $m(A \cap B_t) > \frac{1}{1000}$.

3. Give an example of a sequence of functions $f_i \in L^2(\mathbb{R})$ with $\|f_i\|_{L^2} = 1$, $supp(f_i) \subset [0,1]$, and with $f_i \to 0$ weakly in L^2. Prove that your example has all the desired properties.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Schwartz function, and that $|\hat{f}(\omega)| \leq 1$ and $|\hat{f}(\omega)| \leq |\omega|^{-4}$. Prove that $|f(3) - f(1)| < 1000$.

Work problem 5A or 5B, not both.

5A. Let A be an arc on $\partial\mathbb{D}$ and $f : \mathbb{D} \cup A \to \mathbb{C}$. ($\mathbb{D}$ is the open unit disc in $\mathbb{C}$.) Assume

  (a) f is continuous.

  (b) The restriction of f to $\mathbb{D}$ is holomorphic.

  (c) The restriction of f to A is identically zero.

  Prove that f is identically equal to zero.

5B. Does there exist a sequence of polynomials $P_n(z)$ such that $e^{P_n(z)}$ converges compact uniformly to z? (Prove that there exists such a sequence of polynomials or prove that there does not exist such a sequence of polynomials.)

6. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Prove that $f(z) = az + b$ for complex constants a, b with $a \neq 0$.