Analysis Comprehensive Exam (3 hours) September 2011

No Aids Do all questions.

- 1. (20 points) Suppose that $A \subset [0, 1]$ and $B \subset [0, 1]$ are measurable sets, each of Lebesque measure 1/2. Prove that there exists an $x \in [-1, 1]$ such that $m((A + x) \cap B) \ge 1/10$. Note: Here $(A + x) = \{y \in \mathbb{R}, \text{ such that } y - x \in A\}$. (*Hint:* Use Fubini's theorem).
- 2. Recall the Fourier transform, defined via $\hat{f}(\xi) := \int_{-\infty}^{+\infty} e^{-i\xi \cdot x} f(x) dx$ for functions $f \in L^1(\mathbb{R})$.
 - (a) (15 points) Suppose $f \in L^1(\mathbb{R})$. Prove that $||\hat{f}||_{L^{\infty}} \leq ||f||_{L^1}$. Furthermore, *if in addition* $\hat{f} \in L^1(\mathbb{R})$ then prove that $||f||_{L^{\infty}} \leq ||\hat{f}||_{L^1}$.

For the next question, you may take for granted the following fact: If $f \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{+\infty} (1+|\xi|)^k |\hat{f}(\xi)| d\xi < \infty$$

then f can be modified on a set of measure zero to a function \tilde{f} such that $\tilde{f} \in \mathcal{C}^k$. Furthermore there is a constant M(k) such that:

$$||\tilde{f}||_{\mathcal{C}^k} \le M(k) \int_{-\infty}^{+\infty} (1+|\xi|)^k |\hat{f}(\xi)| d\xi.$$

(b) (15 points) Let $H^m \subset L^2(\mathbb{R})$ be the space of functions for which:

$$\int_{-\infty}^{+\infty} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi < \infty.$$

Assume m > 1/2 + k. Prove that if $f \in H^m$ then f can be modified on a set of measure zero to a function \tilde{f} such that $\tilde{f} \in \mathcal{C}^k(\mathbb{R})$. 3. Consider a function $K(x) \in L^2(\mathbb{R})$. Consider the operator

$$T(f)(x) := (K * f)(x) := \int_{-\infty}^{+\infty} K(x - y)f(y)dy.$$

- (a) (15 points) Prove that $T: L^2 \to L^{\infty}$ is bounded. Find a bound for $||T||_{L^2 \to L^{\infty}}$ in terms of the function K.
- (b) (10 points) Assume in addition that $\hat{K}(\xi)$ is supported in the interval [-U, U], for some U > 0, and $|\hat{K}(\xi)| \leq 1$, $\forall \xi \in [-U, U]$. Prove that for each $m \in \mathbb{N}$, T is bounded from L^2 into \mathcal{C}^m . (*Hint:* You can use the results of the previous exercise without proof).
- 4. Let \mathcal{H} be an infinite-dimensional Hilbert space over the real numbers. Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator.¹
 - (a) (15 points) Prove that at least one of the values ||T||, -||T|| is an eigenvalue of T. You may take for granted that $||T|| = sup_{f \in B}|(Tf, f)|$ for operators of this type.
 - (b) (10 points) Construct an example of a compact but non-selfadjoint operator $T: \mathcal{H} \to \mathcal{H}$ with no non-zero eigenvalues.
- 5. (25 points) Construct a conformal mapping (as a composition of simpler maps) from the infinite strip $|\Im z| < \frac{\pi}{4}$ onto the unit disc which takes 0 to 0. What is the most general such mapping ?
- 6. (25 points) Suppose that Ω is a domain in \mathbb{C} , $\{f_k\}$ is a sequence of analytic functions on Ω , $f_k \to f$ uniformly on compact subsets of Ω , and f has a zero of order N at $z_0 \in \Omega$. Show that there exists $\rho > 0$ such that for k sufficiently large, $f_k(z)$ has exactly N zeros counting multiplicities on $|z z_0| < \rho$.

¹Recall that T is compact when $\overline{T(B)}$ is a compact set in \mathcal{H} , where B is the closed unit ball in \mathcal{H} and $\overline{T(B)}$ is the closure of T(B). Recall also that T is self-adjoint when $T = T^*$.