Department of Mathematics
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Analysis Comprehensive Exam
September 5, 2012

Please be brief but justify your answers, citing relevant theorems.

1. (a) Given two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, with $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, consider their convolution
$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,.$$
Prove that the integral is well-defined for each $x \in \mathbb{R}^n$, and that $f * g$ is bounded.
(b) Furthermore, $f * g$ is continuous, and $\lim_{|x|\to\infty} f * g(x) = 0$.
(c) If, instead, f and g are integrable functions, prove that $f * g(x)$ is well-defined for a.e. x, and agrees almost everywhere with an integrable function.

2. (a) Let ℓ be a bounded linear functional on $L^2(\mathbb{R})$. Prove (directly from the definition) that the function $F(x) = \ell(\mathcal{X}_{[0,x]})$ is absolutely continuous on $[0,1]$. (Here $\mathcal{X}_{[0,x]}$ is the function defined by $\mathcal{X}_{[0,x]}(t) = 1$ if $t \in [0,x]$ and $\mathcal{X}_{[0,x]}(t) = 0$ if $t \notin [0,x]$.)
(b) Use the Riesz representation theorem to find a formula for the derivative $F'(x)$ for a.e. x.

3. (a) Consider the Banach space $L^p([0,1])$ where $1 < p < \infty$. What is the norm of this space? What is the dual of this space? (Just state–no proof needed).
(b) Let $(X, ||\cdot||)$ be a Banach space. Define what it means for a sequence f_n to converge weakly to an element $g \in X$.
(c) Consider a sequence $f_n \in L^p([0,1])$ with $1 < p < \infty$ and assume that $||f_n||_{L^p} \le 1$ and $f_n(x) \to 0$ for almost every $x \in [0,1]$. Prove that f_n converges weakly to 0. (Hint: Use Egorov's theorem.)
(d) Give an example of a sequence $f_n \in L^1([0,1])$ with $||f_n||_{L^1} = 1$ for all n and such that $f_n(x) \to 0$ for almost every $x \in [0,1]$ yet f_n does not converge weakly to 0.

4. (a) Consider the space $L^1([0,2\pi))$. Define the Fourier transform on this space. Define the Fourier transform on $L^2([0,2\pi))$.
(b) Prove that if $f \in L^1([0,2\pi))$ and $\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$ then $f \in L^2([0,2\pi))$.
(c) Prove that if $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$ then f agrees almost everywhere with a continuous function.

5. Let $a \in \mathbb{R}$, $a > 1$. Show that
$$ze^{a-z} = 1$$
has one solution with $|z| < 1$ and that it is real and positive.

6. Suppose that f is analytic in a domain G in the complex plane and not constant. Let D be a disc whose closure is contained in G. Suppose $|f|$ is constant on ∂D. Show that f has at least one zero in D.