DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam ($1\frac{1}{2}$ hours)

May 1998

No aids.
Do all questions.

1. (a) If f is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ when $|z| \leq R/2$.

(b) Suppose f is analytic in a region Ω. Suppose γ is a piecewise smooth closed curve in Ω (not necessarily simple) which does not meet any of the zeros of f. Let p be a nonnegative integer. What is the value of

$$\int_\gamma \frac{f'(z)}{f(z)} z^p dz$$

in terms of the zeros of f?

2. (a) Let f and g be 1-1 analytic mappings from an open connected set $\Omega \subset \mathbb{C}$ onto the unit open disc Δ. Suppose for some point $z_0 \in \Omega$, $f(z_0) = g(z_0) = 0$. What is the relation between f and g?

(b) Let f be a 1-1 analytic map of the unit disc Δ onto the unit square with center 0, satisfying $f(0) = 0$. Show that $f(iz) = if(z)$.

3. (Harnack's Theorem). Let Ω be a connected open subset of $\mathbb{C}$. Let $\{v_n(z)\}_{n=1,2,\ldots}$ be a sequence of real harmonic functions on Ω. Suppose that for all $z \in \Omega$ $v_n(z) \leq v_{n+1}(z)$ for $n = 1, 2, \ldots$ and let

$$v(z) = \lim_{n\to\infty} v_n(z).$$

Show that either $v \equiv +\infty$ on Ω or $v(z)$ is harmonic on Ω.