DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

September 1998

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) Give a complex-analytic proof of the fundamental theorem of algebra: every nonconstant holomorphic polynomial with complex coefficients has a root.

(b) Prove the Casorati-Weierstrass theorem: Suppose f is analytic in the set $0 < |z-a| < R$ and has an essential singularity at a. Then the range of f is dense in $\mathbb{C}$.

2. Verify that

$$w = \frac{(1+z^n)^2 - i(1-z^n)^2}{(1+z^n)^2 + i(1-z^n)^2}$$

transforms the circular sector $0 < |z| < 1$, $0 < \operatorname{Arg} z < \frac{\pi}{n}$ onto the unit disc.

3. Evaluate via residues $\int_0^\infty \frac{x^{a-1}}{1+x} dx$ where $0 < a < 1$.

4. (a) Suppose D is a disc in the complex plane, f is holomorphic in a neighbourhood of $\bar{D}$, and $|f|$ is constant on ∂D. Show that f has at least one zero in D.

(b) Show that any rational function f such that $|f(z)| = 1$ when $|z| = 1$ must have the form $f(z) = Cz^n \prod_{j=1}^{k} \left(\frac{z-a_j}{1-\bar{a}_j z}\right)$ where $n \in \mathbb{Z}$, $|C| = 1$, $a_j \neq 0$, and $|a_j| \neq 1$. (Note that each factor on the right hand side is constant when $|z| = 1$.)