DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

May 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) What is a normal family of analytic functions? What role does this concept play in the proof of the Riemann mapping theorem?

(b) State Picard's theoren on values omitted by entire functions. What role do analytic universal coverings play in its proof?

2. Show that if f is a nonvanishing analytic function on a simply-connected domain $G \subset \mathbb{C}$ then a single-valued analytic branch of $\log f$ can be defined on G.

3. Evaluate $\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x^2 + 2x + 5}\, dx$ via residues ($\lambda > 0$).

4. Let G be a bounded domain in $\mathbb{C}$. Let f be a nonconstant analytic function on G, continuous on $\overline{G}$, with $|f| =$ constant on ∂G. Show that f has at least one zero in G.

5. (a) Determine as small an integer m as you can such that all zeros of $z^6 + z^5 + 100$ lie inside $|z| = m$.

(b) Determine as large an integer n as you can such that all zeros of $z^6 + z^5 + 100$ lie outside $|z| = n$.

6. (a) Find the image of the unit disc under the mapping $f(z) = \frac{1}{2} \log \frac{1+z}{1-z}$.

(b) Show that $|f(z)| \leq \frac{1}{2} \log \frac{1+|z|}{1-|z|}$.