DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

September 7, 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. Suppose that f is analytic in $|z| < R_0$ and that $|a| < R < R_0$. Evaluate

$$\int_\gamma \frac{(R^2 - |a|^2) f(z)}{(z-a)(R^2 - z\bar{a})}\, dz$$

where γ is the circle $|z| = R$ transversed counterclockwise. Hence prove that if $0 < r < R$

$$f(re^{i\theta}) \;=\; \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\phi}) d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2}$$

(Poisson's formula).

2. (a) Show that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right)$ then f is analytic in the whole plane minus the points $0, 1, 2, \ldots$.

(b) Find $f^{(k)}(z)$. (Justify.)

3. Find the image of the upper half plane under the mapping $w = \frac{1-z^\alpha}{1+z^\alpha}$ where $0 < \alpha < 1$ and z^α has its principal value.

4. Suppose f is entire and $|f(z)| \le C_1 + C_2|z|$. What can you say about the form of f? Give a proof.