DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

May 1, 2000

No aids.
Do all questions.
The value of each is indicated. Total =80

1. (a) **[6 marks]**
Evaluate $\int_\gamma \frac{zdz}{z^4-1}$ where γ is the circle $|z-ia|=1$ traversed once counterclockwise and $a>1$.

(b) **[6 marks]**
Evaluate $\int_\gamma \frac{ze^z}{(z-b)^3}dz$ where γ is the unit circle traversed once counterclockwise and $|b|\neq 1$.

(c) **[8 marks]**
Let U be the open unit disc, i.e. $U=\{z\in\mathbb{C} \mid |z|<1\}$. Is $U\backslash\{0\}$ biholomorphic to $\mathbb{C}\backslash\{0\}$? Explain.

2. **[20 marks]**
A family $\mathcal{F}$ of holomorphic functions on a domain Ω is said to be normal if any sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω. Suppose that the family $\mathcal{F}$ fails to be normal. Show that there exists a point $z_0\in\Omega$ such that $\mathcal{F}$ is not normal in any neighbourhood of Ω.

Hint: a compactness argument.

3. **[20 marks]**
Suppose that f and g are meromorphic in $\mathbb{C}$. Suppose that γ is a smooth closed curve in $\mathbb{C}$ (not necessarily simple) such that neither f nor g has any singularities on γ. Suppose also that $|g|<|f|$ on γ. Let

$$\varphi(t)=\int_\gamma \frac{f'(z)+tg'(z)}{f(z)+tg(z)}dz \qquad 0\leq t\leq 1\,.$$

Show that φ is constant.

4. **[20 marks]**
Evaluate $\int_0^\infty \frac{x^\alpha}{x^2+a^2}dx$ via residues where $-1<\alpha<1$ and $a>0$.