DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

May 1, 2000

No aids.
Do all questions.
The value of each is indicated. Total =80

1. (a) [**6 marks**]
Evaluate $\int_\gamma \frac{z dz}{z^4 - 1}$ where γ is the circle $|z - ia| = 1$ traversed once counterclockwise and $a > 1$.

(b) [**6 marks**]
Evaluate $\int_\gamma \frac{ze^z}{(z-b)^3} dz$ where γ is the unit circle traversed once counterclockwise and $|b| \neq 1$.

(c) [**8 marks**]
Let U be the open unit disc, i.e. $U = \{z \in \mathbb{C} \mid |z| < 1\}$. Is $U \backslash \{0\}$ biholomorphic to $\mathbb{C} \backslash \{0\}$? Explain.

2. [**20 marks**]
A family $\mathcal{F}$ of holomorphic functions on a domain Ω is said to be normal if any sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω. Suppose that the family $\mathcal{F}$ fails to be normal. Show that there exists a point $z_0 \in \Omega$ such that $\mathcal{F}$ is not normal in any neighbourhood of Ω.

Hint: a compactness argument.

3. [**20 marks**]
Suppose that f and g are meromorphic in $\mathbb{C}$. Suppose that γ is a smooth closed curve in $\mathbb{C}$ (not necessarily simple) such that neither f nor g has any singularities on γ. Suppose also that $|g| < |f|$ on γ. Let

$$\varphi(t) = \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz \qquad 0 \leq t \leq 1 .$$

Show that φ is constant.

4. [**20 marks**]
Evaluate $\int_0^\infty \frac{x^\alpha}{x^2 + a^2} dx$ via residues where $-1 < \alpha < 1$ and $a > 0$.