DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

September 5, 2000

No aids.
Do all questions.
Questions will be weighted equally.

1. Suppose $\alpha \in \mathbb{C}$, $|\alpha| \neq 1$. Compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

by integrating $(z-\alpha)^{-1}(z-\frac{1}{\alpha})^{-1}$ around the unit circle.

2. Show that the set of biholomorphic maps from the upper half plane onto itself consists precisely of the set of linear fractional transformations of the form $\frac{az+b}{az+d}$ where $a, b, c, d \in \mathbb{R}$ (after cancellation of a common factor) and $ad - bc > 0$.

3. How many zeros does $z^5 + 3z^4 + 12z^3 + 2z - 1$ have in the annulus $1 < |z| < 2$? Explain.

4. Suppose that f is a holomorphic function in a domain $\Omega \subset \mathbb{C}$. Suppose that f has a zero of order k at a point $z_0 \in \Omega$. Show that there is a neighbourhood U of z_0 and a neighbourhood V of $f(z_0)$ such that if $w_0 \in V \backslash \{f(z_0)\}$, the equation $f(z) - w_0 = 0$ has k distinct roots in $U \backslash \{z_0\}$.