DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam ($1\frac{1}{2}$ hours)

Tuesday, September 7, 2004, 2–3:30 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) Define normal family of analytic functions.

(b) Let Ω be a connected open set in $\mathbb{C}$. Let $\{f_n\}$ be a sequence of polynomials in z, each of degree $\leq D$. Suppose that the sequence $\{f_n\}$ converges uniformly on compact subsets of Ω to a function of f. Prove that f is a holomorphic polynomial of degree $\leq D$.

2. (a) State Schwarz's Lemma.

(b) Let f be an analytic mapping from the unit disc $\Delta = \{z \mid |z| < 1\}$ to itself satisfying $f(a) = b$ for some points $a, b \in \Delta$. Prove that

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

Hint: Consider $g := \phi_b \circ f \circ \phi_{-a}$ where

$$\phi_\alpha := \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

3. Let $\{h_n\}$ be a sequence of harmonic functions on a connected open set $\Omega \subset \mathbb{C}$. Suppose that the sequence $\{h_n\}$ converges uniformly on compact subsets of Ω to a function h. Prove that h is harmonic on Ω.