University of Toronto

Faculty of Arts and Science

Final Examinations, April–May 2007

MAT454H1S/1001HS — Complex Analysis

Instructor: Edward Bierstone

Duration — 3 hours

No aids allowed.
Total marks for this paper: 100.
All questions are equal in value.

1.(a) Let U, V denote domains in $\mathbb{C}$ and let $f : U \to V$ be a holomorphic mapping. Suppose that f is *proper* (i.e., $f^{-1}(K)$ is compact, for every compact subset K of V). Prove that $f(U) = V$.

(b) Is the assertion in (a) true if "holomorphic" is replaced by "continuous"? Explain.

2.(a) Let $f(z)$ denote a holomorphic function in $|z| < R$ such that $|f(z)| \le M$. Suppose that $f(z_0) = w_0$, where $|z_0| < R$. Show that

$$\left| \frac{M(f(z) - w_0)}{M^2 - \overline{w_0} f(z)} \right| \le \left| \frac{R(z - z_0)}{R^2 - \overline{z_0} z} \right| .$$

(Hint. First consider the case $f(0) = 0$.)

(b) Show that if $|f(z)| \le 1$ for $|z| < 1$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} .$$

3. Let $\{f_n\}$ be a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ which is bounded uniformly on compact subsets of Ω. Let $\{z_k\}$ be a sequence of distinct points in Ω with $\lim_{k\to\infty} z_k = z_0 \in \Omega$. Assume that $\lim_{n\to\infty} f_n(z_k)$ exists, for all k. Prove that $\{f_n\}$ converges uniformly on compact subsets of Ω.

4. Use residues to show that

$$\int_0^1 \frac{dx}{\sqrt[3]{x^2 - x^3}} = \frac{2\pi}{\sqrt{3}} .$$

5.(a) An *elliptic function* on $\mathbb{C}$ means a doubly-periodic meromorphic function. Show that any *even* elliptic function $f(z)$ can be written in the form

$$f(z) = c \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

(where c is a constant and $\wp(z)$ denotes the Weierstrass $\wp$-function with the same periods), provided that 0 is neither a zero nor a pole of f.

(b) Show that every even elliptic function f can be written $f = R(\wp)$, where R is a rational function.

(c) Show that every elliptic function f can be written $f = R(\wp, \wp')$, where R is rational.

6.(a) Give an example of a meromorphic function on $\mathbb{C}$ that omits *two* values.

(b) Prove *Picard's little theorem for meromorphic functions*: Every meromorphic function on $\mathbb{C}$ that omits three distinct values is constant.