University of Toronto
Faculty of Arts and Sciences
April-May 2009 Examinations
MAT454H1S/1001H1S
3 hours
No Aids Allowed

April 30, 2009 **Final Exam** **C. Pugh**

Problems 1 and 2 are worth ten points, the others are worth 20 points.

As usual, $\mathbb{C}$ is the complex plane, $\widehat{\mathbb{C}}$ is the extended complex plane (the Riemann sphere), $\mathbb{H}$ is the open upper half plane, $\mathbb{D}$ is the open unit disc, and $\mathbb{R}$ is the real number line.

(1) Answer True or False. +2 points for correct answer, −2 points for incorrect answer, 0 points for no answer.
 (a) Every Mobius transformation $M : \mathbb{D} \to \mathbb{D}$ has a fixed point in $\mathbb{D}$.
 (b) The family of all holomorphic functions $f : \mathbb{D} \to \mathbb{H}$ such that $f(0) = i$ is normal.
 (c) There is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that for all $x \in \mathbb{R}$, $\left| f(x) - x^2|\sin x^2| \right| < e^{-|x|^2}$.
 (d) If $f : \Omega \to \mathbb{C}$ is holomorphic and injective then the area of $f(\Omega)$ is $\int_\Omega |f'(z)|^2 \, dxdy$ where $z = x + iy$.
 (e) Every bounded, injective holomorphic function $f : \mathbb{D} \to \mathbb{C}$ extends continuously to the closed disc.

(2) Use complex analysis to calculate the integral

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1} \, dx \ .$$

(3) Find a succession of specific conformal transformations that convert the lens shaped open set $\{z : |z| < 1 \text{ and } |z-1| < 1\}$ to $\mathbb{D}$.

(4) State the Harnack Inequality about positive harmonic functions, and use it (or not) to prove that every nonconstant harmonic function $u : \mathbb{C} \to \mathbb{R}$ is neither bounded below nor bounded above, but rather has $u(\mathbb{C}) = \mathbb{R}$.

(5) Prove that except for the identity map, no holomorphic $f : \mathbb{D} \to \mathbb{D}$ has more than one fixed point.

(6) Suppose $0 \le r < R \le \infty$. Find (with proof) the automorphism group of the annulus $A = \{z : r < |z| < R\}$.

Extra Credit. (Note: I give little *partial* credit on Extra Credit problems.)

(1) Prove or disprove that there exists an injective holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that $f(\mathbb{D})$ is open, dense, and has area $< 1/100$.

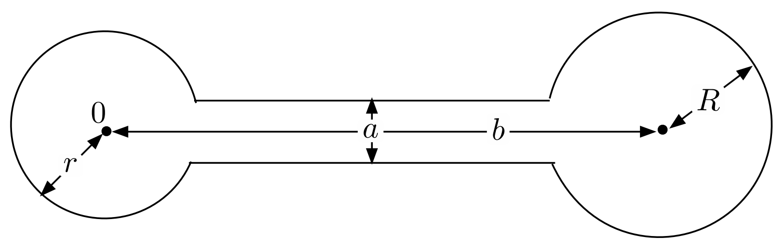
(2) Consider the Riemann map $f : \mathbb{D} \to \Omega$ where Ω is the dumbbell region with parameters a, b, r, R shown in the figure and $f(0) = 0$, $f'(0) > 0$. To what limit, if any, does f converge as

(a) $a \to 0$ and b, r, R are constant.

(b) $b \to \infty$ and a, r, R are constant.

(c) $b \to \infty$ and ab, r, R are constant.

(d) Other distortions such as $a \to 0$ and aR, b, r are constant, etc.



Justify your answers.