Mathematics Department
University of Toronto
Complex Analysis Examination
September 2009
Duration 1.5 hours
No Aids Allowed

Work problem 1A or 1B, not both.

1A. Let A be an arc on $\partial\mathbb{D}$ and $f : \mathbb{D} \cup A \to \mathbb{C}$. ($\mathbb{D}$ is the open unit disc in $\mathbb{C}$.) Assume
(a) f is continuous.
(b) The restriction of f to $\mathbb{D}$ is holomorphic.
(c) The restriction of f to A is identically zero.
Prove that f is identically equal to zero.

1B. Does there exist a sequence of polynomials $P_n(z)$ such that $e^{P_n(z)}$ converges compact uniformly to z? (Prove that there exists such a sequence of polynomials or prove that there does not exist such a sequence of polynomials.)

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Prove that $f(z) = az + b$ for complex constants a, b with $a \neq 0$.

3. Prove that the infinite product

$$\prod_{n=0}^{\infty}(1 + z^{2^n}) = (1 + z)(1 + z^2)(1 + z^4)(1 + z^8)\cdots$$

converges compact uniformly to $1/(1 - z)$ when $|z| < 1$ and diverges when $|z| > 1$.