FINAL EXAMINATION **MAY 2006**

Complex Analysis

Time allowed: 2 hours

Last Name (Please Print): ____________________

First Name: ____________________

INSTRUCTIONS:

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1(20%)	
2(20%)	
3(20%)	
4(20%)	
5(20%)	
TOTAL	

- Equal weight on each problem
- Try as many questions as you can.
- Answer all questions in the space provided.
- Use the back of previous pages if you need more spaces.
- No aids are allowed.

(1) State each of the following carefully and precisely.

(a) The Cauchy Riemann conditions for an analytic function.

(b) The Dirichlet problem concerning a real function harmonic on the open unit disc and (any type of) its solution .

(c) The Cauchy integral formula for the seventh derivative of an analytic function.

(d) The Theorem of Argument concerning the number of zeros and the number of poles.

(e) The Riemann Mapping Theorem.

(2) Let f be a meromorphic fuction (as the quotient of two entire functions). If f has real values on the unit circle, then how are the zeros and the poles of f located? What can you say about the general form of f ?

(3) Consider a formal expression $f(z) = \Sigma_{-\infty}^{\infty} a_n z^n$. What conditions on the coefficients a_n are equivalent to each of the following statements.

(a) f is analytic on the open annulus $\{z: 1 < |z| < 3\}$.

(b) The origin is a pole of order 4 for $f(z) - 3\exp(1/z)$.

(c) The origin is a pole of order 4 for $1/f(z)$.

(d) The origin is an essential singularity for $f(1/z)$.

(e) $f(z)/9 = f(3z)$ for all z in $\mathbf{C} \setminus \{0\}$.

(4) Quick work

(a) Find the image of the open disc $\{z: |z| < 1\}$ under the map $f(z) = (z + i) / (z - i)$.

(b) How many Laurent series of the form $\Sigma_{-\infty}^{\infty} a_n (z - 0)^n$ does the function $(\exp z) / (z^3 - z)$ have? On which open sets, are these expansions valid?

(c) Find two analytic functions f and g defined on $\mathbf{C} \setminus \{0\}$ such that each has an essential singularity at the origin but the product function $f(z)\, g(z)$ has a pole of order 4 at the origin.

(d) How many zeros does the function $10z^5 + 3e^z$ have inside the unit circle? Why?

(e) Suppose the real function $u(x, y)$ and its square $(u(x, y))^2$ are both harmonic on $\mathbf{R}^2$. What can you say about the expression of u? Show your computation.

(**5**) For each of the following cases, find all possible analytic functions $f\colon \mathbf{C} \to \mathbf{C}$.

(**a**) $\operatorname{Im} f = 3y^3 - 2x$.

(**b**) $|f''(z)| < 7$ for all $z \in \mathbf{C}$.

(**c**) The sequence (f(1), f(1/2), f(1/3), f(1/4), f(1/5), ...) is the sequence (2/1, 5/4, 10/9, 17/16, 26/25, ...) with natural pattern.

(**d**) f has a simple zero at each integer and f has no other zeros.

(**e**) f is of the form $f(z) = \alpha z + \beta$ and f sends S into S (i.e., $f(S) \subseteq S$), where S is the open annulus $\{z\colon\ 1 < |z| < 3\}$.