FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATION, APRIL/MAY 2006

MAT 457Y1Y/1000Y
Real Analysis II

Examiner: Professor A. del Junco
Duration: 3 hours

Total marks = 100

Instructions: Do all questions. You may use a previous part of a question to solve a subsequent part even if you have not done the previous part.

Notation: $\mathbb{T}$ denotes the 1-torus $\mathbb{R}/2\pi\mathbb{Z}$.

1. (The parts of this question are **not** related.)
   (a) [10 points] Prove or disprove: there is a norm on $\mathbb{R}^2$ such that the norms of $(0,1)$ and $(1,0)$ are 100 but the norm of $(1,1)$ is 1.
   (b) [15 points] Suppose μ is a finite non-atomic Borel measure on a compact metric space X. (Non-atomic means that $\mu\{x\} = 0$ for all $x \in X$.) Show that for every $\epsilon > 0$ the is a $\delta > 0$ such that $\mu(E) < \epsilon$ for any Borel set E of diameter less than δ.
   (c) [15 points] Suppose μ is a sigma-finite measure on $(X, \mathcal{B})$, $f : X \to [0, \infty)$ is measurable and $\varphi : [0, \infty) \to [0, \infty)$ is an increasing C^1 function such that $\varphi(0) = 0$. Show that

$$\int \varphi(f(x))d\mu(x) = \int_0^\infty \varphi'(t)\mu\{f > t\}dt.$$

2.
   (a) [5 points] Suppose $f \in C^1(\mathbb{T})$. Show that $(f')\widehat{\ }(n) = in\widehat{f}(n)$.
   (b) [10 points] Suppose that $f \in L_1(\mathbb{T})$. Show that f is infinitely differentiable if and only if $\lim_{|n|\to\infty} n^k \widehat{f}(n) = 0$ for every positive integer k.

3. Suppose E is a subspace of $C[0,1]$ which is closed as a subspace of $L_2[0,1]$. Prove the following.
   (a) [5 points] There is a constant C such that

$$\|f\|_2 \le \|f\|_\infty \le C\|f\|_2,$$

   and E is closed in $C[0,1]$. (Use the closed graph theorem.)

(b) [5 points] For every $x \in [0,1]$ there is a $g_x \in E$ such that $f(x) = \langle f, g_x \rangle$ for every $f \in E$ and $\|g_x\|_2 \leq C$.

(c) [5 marks] If e_j is an orthonormal basis for E then $\sum_j |e_j(x)|^2 \leq C^2$ for every $x \in [0,1]$.

(d) [5 points] E is finite-dimensional, in fact $\dim E \leq C^2$.

4. Suppose f is a real-valued function defined on $[0,1]$. Recall that the arclength of f is defined as

$$\sup \sum_{i=0}^{n-1} \|(t_i, f(t_i)) - (t_{i+1}, f(t_{i+1}))\|_2,$$

where the supremum extends over all partitions $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of $[0,1]$ and $\|\cdot\|_2$ denotes the usual Euclidean norm. Prove the following.

(a) [5 points] If f is increasing on $[0,1]$, $f(0) = 0$ and $f(1) = 1$ then the arclength of f lies between $\sqrt{2}$ and 2.

(b) [10 points] If $f(x) = \mu(0,x]$ where μ is a singular (with respect to Lebesgue measure) Borel probability measure on $[0,1]$ then the arclength of f is 2.

5. Suppose $f_1, f_2, \ldots$ is an orthonormal sequence in $L_2[0,1]$. Prove that

$$A_n = \frac{1}{n} \sum_{i=1}^{n} f_i \to 0$$

μ-a.e. by filling in the details of the outline below.

(a) [7 points] Let $E_n = \{|f_n| > n^{\frac{2}{3}}\}$. Then $\mu(E_n) \leq \frac{1}{n^{\frac{4}{3}}}$. For almost every x there is an $N(x)$ such that $x \in E_n$ for $n > N(x)$.

(b) [5 points] $\|A_N\|_2^2 = \frac{1}{N}$. Let $N_j = [j \log^2 j]$ ($[x]$ denotes the greatest integer less than or equal to x). $\sum_{j>0} |A_{N_j}|^2$ is finite a.e. and $A_{N_j} \to 0$ a.e. as $j \to \infty$.

(c) [8 points] For $N_j \leq N < N_{j+1}$

$$|A_N(x)| \leq |A_{N_j}(x)| + \frac{1}{N_j} \sum_{n=N_j}^{N_{j+1}-1} |f_n(x)|.$$

For sufficiently large N use the result in part (a) to estimate the second term above and conclude that $\lim_N A_N(x) = 0$.