DEPARTMENT OF MATHEMATICS
University of Toronto

Real Analysis Exam (2 hours)

May 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) What is the dual of $L^3(\mathbb{R})$?

(b) Show that the dual of ℓ^∞ ("bounded functions on the positive integers") is *not* ℓ^1 ("summable functions on the positive integers"), by exhibiting an element of the dual that is not in ℓ^1.

2. Let f be a non-negative integrable function on $\mathbb{R}$ (with Lebesgue measure), let μ be Lebesgue measure on $\mathbb{R}^2$, and show that

$$\mu\big(\{(x,y) : 0 \le y \le f(x)\}\big) = \mu\big(\{(x,y) : 0 < y < f(x)\}\big) = \int f(x)\,dx.$$

3. For some measures, $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, $L^r \supset L^s$, and for some measures, L^r can never contain L^s unless $r = s$. Find and explain examples of each phenomenon and/or necessary and/or sufficient conditions.

4. Let $\{\delta_n\}$ be a sequence of positive numbers, and $\{\phi_n\}$ an orthonormal set in an infinite-dimensional Hilbert space $\mathcal{H}$. Set

$$S = \{x = \sum_{n=1}^{\infty} a_n \phi_n \in \mathcal{H} :\mid a_n \mid \le \delta_n\}.$$

Prove S is compact if and only if $\sum \delta_n^2 < \infty$. (In the case $\delta_n = \frac{1}{n}$, S is called the "Hilbert cube").

5. Find the maximum value of $\int_{-1}^{1} x^3 g(x)dx$, for measurable functions $g(x)$ satisfying

$$\int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} x^2 g(x)dx = 0,$$

and $\int_{-1}^{1} \mid g(x) \mid^2 dx = 1$.

6. Evaluate the derivative and the second derivative of the Heaviside function H on $\mathbb{R}$, in the sense of distributions; the Heaviside function is:

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$