

DEPARTMENT OF MATHEMATICS
University of Toronto

Real Analysis Exam (2 hours)

May 1, 2000

No aids.

Do all questions.

The value of each is indicated.

For each part provide a complete justification of your claims.

Total = 80.

1. [$5 \times 4 = 20$ marks]

Prove or give counterexample:

- (a) Any open set $U \subset (0, 1)$ which includes all rationals in the unit interval has full Lebesgue measure: $m(U) = 1$.
- (b) The Banach space dual of $L^\infty([0, 1])$ is $L^1([0, 1])$.
- (c) If $f_n \rightarrow f$ in $L^1(\mathbf{R})$, then f_n converges pointwise a.e.
- (d) A non-decreasing function $f : \mathbf{R} \rightarrow \mathbf{R}$ is differentiable a.e.

2. [$10 \times 2 = 20$ marks]

Let $\psi(x) \geq 0$ be a smooth, compactly supported function satisfying $\int_{-\infty}^{\infty} \psi(x) dx = 1$. Given any (real-valued function) $f \in L^1(\mathbf{R})$ define its convolution with ψ by

$$g(x) = \int_{\mathbf{R}} f(y)\psi(x-y)dy.$$

- (a) Prove $g(x)$ is differentiable everywhere.
- (b) Prove $\|g\|_1 \leq \|f\|_1$ for real $f \in L^1(\mathbf{R})$ and discuss the cases of equality.

3. [$10 \times 2 = 20$ marks]

- (a) Given a closed subspace $K \subset H$ of a Hilbert space H , prove that each $h \in H$ can be uniquely expressed as $h = k + j$, where $k \in K$ and j is orthogonal to K .
- (b) Among $f \in L^2[0, 1]$ satisfying $\|f\|_2 = 1$, which function(s) maximize $\int_0^1 xf(x) dx$?

4. (a) [5 marks]

Compute the Fourier transform of the Gaussian $g(x) = e^{-x^2/2}$.

(b) [5 marks]

Compute the Fourier transform of the error function

$$\Phi(x) = \int_{-\infty}^x e^{-t^2/2} dt.$$

(c) [10 marks]

Construct an elementary function whose Fourier transform is continuous but fails to be differentiable.