DEPARTMENT OF MATHEMATICS
University of Toronto

Real Analysis Exam (2 hours)

Tuesday, September 3, 2002, 1–3 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. Prove or find a counterexample for each of the following statements. Here, all integrals are defined by the Lebesgue measure m on the real line.

(a) If A is a measurable set then there exists a sequence of open sets $U_n \subseteq A$ such that $m(A) = \lim_{n\to\infty} m(U_n)$.

(b) If $\int_{-1}^{1} x^n f(x)dx = 0$ for each $n = 0, 1, \ldots$ where f is a bounded and measurable function on $[-1, 1]$ then $f = 0$ a.e.

(c) $f(b) - f(a) = \int_a^b f'(x)dx$ for any continuous and increasing function f.

2. (a) Prove that $f \in L^q(\mathbb{R})$ for any q such that $p_1 \leq q \leq p_2$ whenever $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$. Here, $0 < p_1 < p_2$ and the L^p spaces are relative to the Lebesgue measure on $\mathbb{R}$.

(b) Show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ whenever $p \neq q$.

Which of the above statements becomes false if the real line is replaced by a finite interval $[a, b]$. Explain.

3. Let $C[0, 1]$ denote the space of continuous functions on $[0, 1]$ equipped with the sup-norm, and let K be a continuous function on $[0, 1] \times [0, 1]$. Suppose that L: $C[0, 1] \to C[0, 1]$ is defined by $Lf = g$ if and only if $g(y) = \int_0^1 K(x, y)f(x)dx$ for all $y \in [0, 1]$. Prove that $\{Lf_n\}$ contains a convergent subsequence in $C[0, 1]$ for any bounded sequence $\{f_n\}$ in $C[0, 1]$.

4. Find the maximum value of $\int_{-1}^{1} x^5 f(x)dx$ among all Lebesgue measurable functions f on $[-1, 1]$ such that $\int_{-1}^{1} f^2 dx = 1$, and $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} xf(x)dx = \int_{-1}^{1} x^2 f(x)dx = 0$.