DEPARTMENT OF MATHEMATICS
University of Toronto

**Real Analysis Exam (2 hours)**

*Tuesday, September 2, 2003, 1–3 p.m.*

No aids.
Do all questions.
Questions will be weighted equally.

**1.** (a) Let $E$ be a normed real vector space, $x_0 \in E$. Prove there exists a linear functional $\phi$ on $E$ so that $\phi(\alpha x_0) = -3\alpha$ for all $\alpha \in \mathbb{R}$.

(b) Assume $E$ has an inner product. Use the inner product to write your $\phi$ in an explicit manner.

**2.** Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx = 0.$$

**3.** Let $X$ be a Banach space and $A : X \to X$ a bounded operator. Recall that $\mathcal{L}(X, X)$ is the Banach space of bounded linear operators from $X$ to $X$ with the norm induced by the norm on $X$.

a) Fix $t \in \mathbb{R}$. Construct

$$e^{tA} \in \mathcal{L}(X, X).$$

(That is, define an operator $B$ that is the most sensible definition of $e^{tA}$ that you can think of and prove that $B \in \mathcal{L}(X, X)$.)

b) Given $x_0 \in X$ we define a path $x(t) \in X$ for $t \in \mathbb{R}$ by

$$x(t) = e^{tA} x_0.$$

Prove that

$$\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$

exists in $X$ and call this limit "$dx/dt$ at time $t$". Prove that at each time $t$

$$\frac{dx}{dt}(t) = Ae^{tA}x_0 = Ax(t).$$

**4.** Let $H$ be a Hilbert space and $A : H \to H$ be a bounded linear operator. The point spectrum of $A$ is:

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid Ax = \lambda x, \quad \text{for some } x \in H,\ x \neq 0\}$$

Prove or disprove:

$$\sup\{|\lambda| :\ \lambda \in \sigma(A)\} = \|A\|.$$