

DEPARTMENT OF MATHEMATICS

University of Toronto

Real Analysis Exam (2 hours)

Friday, May 7, 2004, 1–3 p.m.

All problems have equal weight.

1. Prove or find a counterexample for each of the following statements.
Here μ denotes the Lebesgue measure on the real line.

(a) Let $\{\phi_n\}$ denote a sequence of functions on $[0, 1]$ such that

$$\int_0^1 |\phi_n| d\mu \leq C$$

for some constant C , and

$$\lim_{n \rightarrow \infty} \int_E \phi_n d\mu = 0 \text{ for each measurable subset } E \text{ of } [0, 1].$$

Then, $\lim_{n \rightarrow \infty} \int_0^1 f \phi_n d\mu = 0$ for each bounded and measurable function f on $[0, 1]$.

(b) If $\{f_n\} \subset L^2([0, 1])$ and $\lim_{n \rightarrow \infty} \int_0^1 f_n^2 d\mu = 0$, then $\lim_{n \rightarrow \infty} f_n(x) = 0$ for almost all $x \in [0, 1]$.

2. Let μ denote a finite Borel measure on R . Evaluate the following limits:

$$\lim_{t \rightarrow \infty} \int_R f(tx) d\mu(x)$$

$$\lim_{t \rightarrow 0} \int_R f(tx) d\mu(x),$$

where f denotes a continuous function on R with compact support.
Justify your answers.

3. Let H denote a real Hilbert space with an inner product $\langle x, y \rangle$. Consider the function $Q(x, y) = \langle x, Ay \rangle$ on $H \times H$ for some linear transformation A on H .

- (a) Show that Q is continuous if and only if A is continuous.
 (b) If Q is continuous show that

$$\sup \frac{|Q(x, y)|}{\|x\| \|y\|} = \sup \frac{\|A(x)\|}{\|x\|}, \quad x \neq 0, y \neq 0.$$

4. Let M denote a closed linear subspace of a Banach space X and let x be any point of X not in M .

- (a) Show that $\inf\{\|x - y\| : y \in M\} = \delta > 0$
 (b) Show that there exists a bounded linear functional f on X such that $f(x) = \|x\|$ and $f(y) = 0$ for all $y \in M$.