University of Toronto

Department of Mathematics

Real Analysis Examination

Monday, May 2, 2005, 1–3 p.m.

Duration 2 hours

No aids allowed.
All questions are equal in value.
In case a problem contains an assertion the problem is to prove that assertion.

1. (a) A Hilbert space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.

(b) Suppose $\mathcal{H}$ is any separable infinite-dimensional Hilbert space. Then $\mathcal{H}$ has a family of closed subspaces $\{E_t : t \in [0,1]\}$ such that E_s is a strict subspace of E_t for all $0 \leq s < t \leq 1$.

2. In this problem (X, μ) is any finite measure space.

(a) Suppose $f_n \in L_1(\mu)$ and $\|f_n\|_1 \leq 1$ for all n. Prove or disprove: $f_n/n \to 0$ almost everywhere as $n \to \infty$.

(b) Suppose f_n are measurable functions on X with values in $(-\infty, \infty)$. (The f_n are not assumed to be integrable.) Then there are constants $c_n > 0$ such that $c_n f_n \to 0$ almost everywhere.

3. In this problem X is a Banach space and $P : X \to X$ is a (not necessarily continuous) linear map such that $P^2 = P$. Let $R = P(X)$ denote the range of P and N the kernel of P.

(a) X is the direct sum of P and N, that is $X = P + N$ and $P \cap N = \{0\}$.

(b) P is continuous if and only if R and N are closed.

4. In this problem g is a continuous 2π-periodic function on $\mathbb{R}$. For $n \in \mathbb{Z}$ the n-th Fourier co-efficient of g is

$$\widehat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

and the convolution $g * g$ is defined by

$$g * g(x) = \frac{1}{2\pi} \int_0^{2\pi} g(x-y) g(y) dy.$$

State precisely any facts about the Fourier transform which you use in your arguments.

(a) $g \in C^\infty(\mathbb{R})$ if and only if for each $k > 0$ there is a constant C_k such that $|\widehat{g}(n)| \leq C_k |n|^{-k}$.

(b) If $g * g \in C^\infty(\mathbb{R})$ then $g \in C^\infty(\mathbb{R})$.