

University of Toronto
Department of Mathematics
Real Analysis Examination

Tuesday, September 6, 2005, 1–3 p.m.
Duration 2 hours

No aids allowed.

All questions are equal in value.

1. Let μ denote Lebesgue measure on the unit interval $[0, 1]$ and \mathcal{B} the σ -algebra of Lebesgue measurable subsets of $[0, 1]$. Given sets $A, B \in \mathcal{B}$ define $d(A, B) = \mu(A \Delta B)$. ($A \Delta B$ denotes the symmetric difference.) Show that d is a pseudo-metric on \mathcal{B} and hence defines a metric on the space $\bar{\mathcal{B}}$ of equivalence classes of sets in \mathcal{B} modulo null sets. Show that this metric is complete. Give an example to show that it is not compact.
2. Suppose x_n is a sequence of vectors in a Hilbert space \mathcal{H} which converges weakly to a limit x .
 - (a) Show that $\|x_n\|$ is bounded.
 - (b) Show that there is a subsequence $\{x_{n_i}\}$ such that $\frac{1}{N} \sum_{i=1}^N x_{n_i}$ converges in norm to x . Suggestion: show that without loss of generality one may take $x = 0$. After x_{n_j} has been chosen for $j < i$ choose x_{n_i} so that $\langle x_{n_i}, x_{n_j} \rangle < 2^{-i}$ for all $j < i$. Use this to estimate $\|\sum_{i=1}^N x_{n_i}\|$.
 - (c) Use part (b) to show that any convex norm-closed subset of \mathcal{H} is weakly closed.
3. Suppose α is any irrational number and define a transformation T from the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$ to itself by $Tx = x + \alpha$. The addition is in \mathbb{R}/\mathbb{Z} , that is Tx is the fractional part of $x + \alpha$ as a transformation of $[0, 1)$. Let μ denote Lebesgue measure on \mathbb{T} . If $f \in L_2(\mu)$ and $f = f \circ T$ show that f is (almost everywhere) equal to a constant function. Hint: expand f in a Fourier series.
4. Show that any norm-closed subspace of a normed vector space is weakly closed. By considering $c_0(\mathbb{N}) \subset l_\infty(\mathbb{N})$ show that a norm-closed subspace of the dual of a normed vector space need not be closed in the weak-* topology.