University of Toronto

Department of Mathematics

Real Analysis Examination

Tuesday, September 6, 2005, 1–3 p.m.
Duration 2 hours

No aids allowed.
All questions are equal in value.

1. Let μ denote Lebesgue measure on the unit interval $[0,1]$ and $\mathcal{B}$ the σ-algebra of Lebesgue measurable subsets of $[0,1]$. Given sets $A, B \in \mathcal{B}$ define $d(A,B) = \mu(A\Delta B)$. ($A\Delta B$ denotes the symmetric difference.) Show that d is a pseudo-metric on $\mathcal{B}$ and hence defines a metric on the space $\bar{\mathcal{B}}$ of equivalence classes of sets in $\mathcal{B}$ modulo null sets. Show that this metric is complete. Give an example to show that it is not compact.

2. Suppose x_n is a sequence of vectors in a Hilbert space $\mathcal{H}$ which converges weakly to a limit x.
   (a) Show that $\|x_n\|$ is bounded.
   (b) Show that there is a subsequence $\{x_{n_i}\}$ such that $\frac{1}{N}\sum_{i=1}^{N} x_{n_i}$ converges in norm to x. Suggestion: show that without loss of generality one may take $x = 0$. After x_{n_j} has been chosen for $j < i$ choose x_{n_i} so that $\langle x_{n_i}, x_{n_j}\rangle < 2^{-i}$ for all $j < i$. Use this to estimate $\|\sum_{i=1}^{N} x_{n_i}\|$.
   (c) Use part (b) to show that any convex norm-closed subset of $\mathcal{H}$ is weakly closed.

3. Suppose α is any irrational number and define a transformation T from the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1)$ to itself by $Tx = x + \alpha$. The addition is in $\mathbb{R}/\mathbb{Z}$, that is Tx is the fractional part of $x + \alpha$ as a transformation of $[0,1)$. Let μ denote Lebesgue measure on $\mathbb{T}$. If $f \in L_2(\mu)$ and $f = f \circ T$ show that f is (almost everywhere) equal to a constant function. Hint: expand f in a Fourier series.

4. Show that any norm-closed subspace of a normed vector space is weakly closed. By considering $c_0(\mathbb{N}) \subset l_\infty(\mathbb{N})$ show that a norm-closed subspace of the dual of a normed vector space need not be closed in the weak-$*$ toplogy.