University of Toronto

Department of Mathematics

Real Analysis Examination

Tuesday, September 5, 2006, 1-3:00 p.m.

Duration 2 hours

Instructions: Do all questions. You may use a previous part of a question to solve a subsequent part even if you have not done the previous part. Do not be concerned if you can't do all the problems. Good luck

1. [25 marks] Quickies
   (a) Prove that if T is a linear transformation of a Hilbert space $\mathcal{H}$ into itself such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$ then T is bounded.
   (b) Suppose C is a convex subset with non-empty interior in a normed vector space X. Show that the interior of C is dense in C. (Hint: draw a picture.)
   (c) Suppose $\{a_i\}$ is a sequence in l_2 and a_i is never zero. Show that there is a $\{b_i\} \in l_1$ such that $\{\frac{b_i}{a_i}\}$ is not in l_2. (Hint: restate this as a question about a linear operator between Banach spaces.)
2. [15 marks]
   (a) Suppose $f, g \in L^2(\mathbb{R})$. Show that the convolution $f * g$ is an everywhere defined continuous function.
   (b) Suppose $f \in L^1(\mathbb{R})$ and is not equal a.e. to the zero function. Show that $f * f$ is not identically zero. (Hint: use properties of the Fourier transform.)
   (c) Deduce from (b) that if $A, B \subset \mathbb{R}$ are subsets of positive measure then the set $A+B$ contains a non-empty open interval. (Hint: it is easy to reduce to the case when $A = B$ and A has finite measure.)
3. [15 marks] Suppose $1 \le p < \infty$ $f_n, f \in L_p[0,1]$, $f_n \to f$ almost everywhere and $\|f_n\|_p \to \|f_p\|$. Prove or disprove: $\|f_n - f\|_p \to 0$
4. [15 marks] Let m denote Lebesgue measure on $\mathbb{R}^2$ and suppose that $A \subset \mathbb{R}^2$ with $m(A) < \infty$.
   (a) Prove that $m(A\Delta(A+x)) \to 0$ as $x \to 0$ in $\mathbb{R}^2$. Here Δ denotes the symmetric difference of set. (Hint: this is clear if $A = I \times J$ where I and J are bounded intervals in $\mathbb{R}$. Proceed by an approximation argument.)
   (b) Suppose $m(A) > 0$. Use part (a) to show that there is an $\epsilon > 0$ such that for each $\delta < \epsilon$, A contains the vertices of some square of side δ. (Hint: show that the intersection of four suitable translates of A has positive measure, and hence is non-empty.)