FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL 30, 2007

Mat457Y1Y/1000YY
Real Analysis II

Examiner: Professor G. Forni
Duration: 3 hours

No calculators or other aids are allowed.

Attempt **ALL** *questions.*
Total: 100 points.
All questions are of equal value.

(I) Let m be the Lebesgue measure on $[0,1]$.

(a) Does there exists a sequence of measurable functions (f_n) in $L^1([0,1],m)$ such that $f_n(x) \to f(x)$ (as $n \to +\infty$) for m-almost all $x \in X$ and $\int_{[0,1]}(f_n - f)dm \to 0$ but $\int_{[0,1]} |f_n - f| dm = 1$ for all $n \in \mathbf{N}$?

(b) For which $\alpha > 0$ does there exists a sequence (E_n) of Borel subsets of $[0,1]$ such that $m(E_n) \le 1/n^\alpha$ for all $n \in \mathbf{N} \setminus \{0\}$ and

$$m\{x \in [0,1] \,|\, \text{the set } \{n \in \mathbf{N} | x \in E_n\} \text{ is infinite}\} \; > \; 0\,?$$

(II) For any $p \in [1,+\infty)$, let B_p be the closed unit ball in $L^p[1,+\infty)$, that is, the set of functions $f \in L^p[1,+\infty)$ such that $\|f\|_p \le 1$. Let

$$I(f) := \int_{[1,+\infty)} \frac{f(x)}{x}\, dx \quad \text{and} \quad M_p := \sup\{I(f) | \, f \in B_p\}\,.$$

(a) Find the value of M_p for all $p \in [1,+\infty)$.

(b) For what $p \in [1,+\infty)$, if any, there exists $f \in B_p$ such that $I(f) = M_p$?

(III)

(a) Let $C([0,1])$ be the space of continuous functions on the closed interval $[0,1]$. Prove that there is no norm on $C([0,1])$ such that the convergence in norm is equivalent to the point-wise convergence in $C([0,1])$.

(*b*) Prove that there is no norm on $L^\infty([0,1])$ such that the convergence in norm is equivalent to the convergence in measure in $L^\infty([0,1])$.

(*IV*) Let $\ell^1(\mathbf{Z})$ be the space of summable bi-infinite sequence of complex numbers and let $C(S^1)$ be the space of continuous functions on the circle $S^1 = \mathbf{R}/\mathbf{Z}$, that is, the space of periodic functions (of period 1) on $\mathbf{R}$, endowed with the uniform norm.

(*a*) Prove that the map $T : \ell^1(\mathbf{Z}) \to C(S^1)$ defined as

$$T((a_n)) = \sum_{n\in\mathbf{Z}} a_n \exp(2\pi i n t)\,, \quad (a_n) \in \ell^1(\mathbf{Z})\,,$$

is a bounded linear operator.

(*b*) Prove that there exists a function $f \in C(S^1)$ such that the sequence (f_n) of its Fourier coefficients, defined as

$$f_n = \int_{S^1} f(t) \exp(-2\pi i n t) dt\,, \quad n \in \mathbf{Z},$$

does not belong to $\ell^1(\mathbf{Z})$.

(*V*) For any complex measure μ on the circle $S^1 = \mathbf{R}/\mathbf{Z}$, let

$$\hat{\mu}(n) = \int_{S^1} \exp(-2\pi i n t)\, d\mu(t)\,,$$

denote its Fourier coefficients.

(*a*) Prove that, if $\hat{\mu}(n) \to 0$ as $n \to +\infty$, then for every bounded Borel function f on S^1

$$\int_{S^1} \exp(-2\pi i n t)\, f(t) d\mu(t) \to 0 \quad \text{as } n \to +\infty\,.$$

(*b*) Prove that, if $\hat{\mu}(n) \to 0$ as $n \to +\infty$, then $\widehat{|\mu|}(n) \to 0$ as $n \to +\infty$ (where $|\mu|$ denotes the total variation of μ) and that, as a consequence, $\hat{\mu}(n) \to 0$ as $n \to -\infty$.