University of Toronto
Faculty of Arts and Sciences
April 2009
Math 1000/457 Final Exam

Duration: 3 hours.

You may use the attached list of formulas. No other aids are allowed during the exam.

Problems 1-3 are each worth 40 points. Each problem has several parts.

1. (Measure theory)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an L^1 function with $\|f\|_{L^1} \leq 1$. Define $g : \mathbb{R} \to \mathbb{R}$ by the formula:

$$g(x) := \int_{-\infty}^{\infty} \frac{1}{1 + |x-y|^2} f(y) dy.$$

a. (10 points) Prove that $\lim_{x \to +\infty} g(x) = 0$.

b. (15 points) Prove that $g(x)$ is continuous.

c. (15 points) Prove that there exists a point x with $|x| < 100$ and $|g(x)| < 1$.

2. (Functional analysis)

a. (20 points) Let B be any Banach space. Let v_n be a sequence of vectors in B. Suppose that v_n converges strongly to w, and that v_n converges weakly to u. Prove that $u = w$.

b. (10 points) Find the maximum of $\int_0^1 x^2 g(x) dx$ among measurable $g : [0,1] \to \mathbb{R}$ with $\int_0^1 |g(x)|^2 dx = 1$.

c. (10 points) Find the maximum of $\int_0^1 x^2 g(x) dx$ among measurable $g : [0,1] \to \mathbb{R}$ with $\int_0^1 |g(x)|^2 dx = 1$ and $\int_0^1 g(x) dx = 0$.

3. (Fourier analysis)

Suppose that f is a Schwartz function on the real line, that f is supported in the interval $[-1,1]$, and that $|f(x)|$ and $|f'(x)|$ are at most 1 for every $x \in [-1,1]$.

a. (10 points) Prove that $|\hat{f}(\omega)| \leq 100\omega^{-1}$ for every $\omega \in \mathbb{R}$.

b. (15 points) Prove that

$$\int_{-\infty}^{\infty} |\omega|^2 |\hat{f}(\omega)|^2 d\omega \leq 100.$$

c. (15 points) Let $I_N f$ be the partial Fourier integral defined by

$$I_N f(x) := \int_{-N}^{N} e^{2\pi i\omega x} \hat{f}(\omega) d\omega.$$

Using part b., prove that $I_N f$ approximates f in the sense that

$$|I_N f(x) - f(x)| \leq 10^4 N^{-1/2} \text{ for every } x \in \mathbb{R}.$$