Math 1000 Fall Semester Final

You have three hours to complete the test.

Problems 1-4 are each worth 25 points.

If you work on a problem but you can't solve it, write down what you tried and say why it doesn't work.

1. Let $A, B \subset \mathbb{R}$ be measurable sets with finite measure. For $t \in \mathbb{R}$, let B_t denote the translation of B by t.

Prove that $\lim_{t\to 0} m(A \cap B_t) = m(A \cap B)$.

2a. Give an example of a subspace $S \subset L^2(\mathbb{R})$ which is not closed. (Prove that your subspace is not closed.)

2b. Let $T = \{f \in L^2(\mathbb{R}) | \int_0^2 f(x)dx = 0\}$. Prove that T is a closed subspace.

2c. What is $T^\perp$? (For most of the credit, write down a non-zero function in $T^\perp$.)

3. Let f, g be functions in $L^2(\mathbb{R})$. We define the convolution $f * g$ by the formula

$$f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy. \qquad (*)$$

3a. For each x, prove that the integral on the right-hand side of $(*)$ is well-defined, and check that $|f * g(x)| \leq \|f\|_{L^2}\|g\|_{L^2}$.

3b. Prove that $\lim_{x\to+\infty} f * g(x) = 0$. (Hint: Approximate f, g by nicer functions.)

4. Let H be a Hilbert space with orthonormal basis $\{e_k\}_{k=1}^\infty$. Define subsets $A, B \subset H$ as follows:

$$A = \{f \in H \text{ such that } \sum_{k=1}^\infty |(f, e_k)| \leq 1\}.$$

$$B = \{f \in H \text{ such that } \sum_{k=1}^\infty |(f, e_k)| \geq 1\}.$$

Is A closed? Is B closed? Prove your answers.

Extra credit. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be an L^2 function. Suppose that f is supported in $[-R, R]$ for some $R > 1$.

Define $Af(x) = \int_x^{x+1} f(y)dy$.

Prove that $\|Af\|_1 \leq 10R^{1/2}\|f\|_2$.