Real Analysis Comprehensive Exam
(2 hours)
September 2011

No Aids

1. (20 points) Suppose that $A \subset [0,1]$ and $B \subset [0,1]$ are measurable sets, each of Lebesque measure $1/2$. Prove that there exists an $x \in [-1,1]$ such that $m((A+x)\bigcap B) \geq 1/10$. *Note:* Here $(A+x) = \{y \in \mathbb{R}, \text{such that } y - x \in A\}$. (*Hint:* Use Fubini's theorem).

2. Recall the Fourier transform, defined via $\hat{f}(\xi) := \int_{-\infty}^{+\infty} e^{-i\xi\cdot x} f(x)dx$ for functions $f \in L^1(\mathbb{R})$.

   (a) (15 points) Suppose $f \in L^1(\mathbb{R})$. Prove that $||\hat{f}||_{L^\infty} \leq ||f||_{L^1}$. Furthermore, *if in addition* $\hat{f} \in L^1(\mathbb{R})$ then prove that $||f||_{L^\infty} \leq ||\hat{f}||_{L^1}$.

   For the next question, you may take for granted the following fact: If $f \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{+\infty} (1+|\xi|)^k |\hat{f}(\xi)| d\xi < \infty$$

   then f can be modified on a set of measure zero to a function $\tilde{f}$ such that $\tilde{f} \in \mathcal{C}^k$. Furthermore there is a constant $M(k)$ such that:

$$||\tilde{f}||_{\mathcal{C}^k} \leq M(k) \int_{-\infty}^{+\infty} (1+|\xi|)^k |\hat{f}(\xi)| d\xi.$$

   (b) (15 points) Let $H^m \subset L^2(\mathbb{R})$ be the space of functions for which:

$$\int_{-\infty}^{+\infty} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi < \infty.$$

   Assume $m > 1/2 + k$. Prove that if $f \in H^m$ then f can be modified on a set of measure zero to a function $\tilde{f}$ such that $\tilde{f} \in \mathcal{C}^k(\mathbb{R})$.

3. Consider a function $K(x) \in L^2(\mathbb{R})$. Consider the operator

$$T(f)(x) := (K * f)(x) := \int_{-\infty}^{+\infty} K(x-y)f(y)dy.$$

(a) (15 points) Prove that $T : L^2 \to L^\infty$ is bounded. Find a bound for $||T||_{L^2 \to L^\infty}$ in terms of the function K.

(b) (10 points) Assume in addition that $\hat{K}(\xi)$ is supported in the interval $[-U, U]$, for some $U > 0$, and $|\hat{K}(\xi)| \leq 1$, $\forall \xi \in [-U, U]$. Prove that for each $m \in \mathbb{N}$, T is bounded from L^2 into $\mathcal{C}^m$. (*Hint:* You can use the results of the previous exercise without proof).

4. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space over the real numbers. Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator.[1]

(a) (15 points) Prove that at least one of the values $||T||, -||T||$ is an eigenvalue of T. You may take for granted that $||T|| = sup_{f \in B}|(Tf, f)|$ for operators of this type.

(b) (10 points) Construct an example of a compact but non-self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ with no non-zero eigenvalues.

[1] Recall that T is compact when $\overline{T(B)}$ is a compact set in $\mathcal{H}$, where B is the closed unit ball in $\mathcal{H}$ and $\overline{T(B)}$ is the closure of $T(B)$. Recall also that T is self-adjoint when $T = T^*$.