UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

APRIL/MAY EXAMINATIONS 2006

Math 1300YY / 427H1S Topology — Final Exam

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Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve 2 of the 3 problems in part A and 4 of the 6 problems in part B. Each problem is worth 17 points, to a maximal total grade of 102. If you solve more than the required 2 in 3 and 4 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones!

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve 5 of the 6 problems in part B, do not solve anything in part A. Each problem is worth 20 points. If you solve more than the required 5 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones!

Duration. You have 3 hours to write this exam.

Allowed Material. None.

Special Request. While it is not *required,* it will be immensely helpful if you could stay after the exam for a post mortem class discussion.

Good Luck!

Part A

Problem 1. Let X be a topological space.

1. Define the product topology on $X \times X$.
2. Prove that if X is connected, so is $X \times X$.

Problem 2. Let X be a topological space, let $C(X, I)$ be the set of all continuous functions on X with values in the unit interval $I = [0, 1]$, and let $\iota : X \to I^{C(X,I)}$ be defined by $\iota(x)_f := f(x)$ for $x \in X$ and for $f \in C(X, I)$.

1. Prove that ι is continuos.
2. Define the phrase "X is completely regular ($T_{3.5}$)".
3. Prove that if X is completely regular then ι is one to one.
4. Prove that if X is completely regular then ι is a homeomorphism of X to $\iota(X)$.

Problem 3.

1. Define the phrase "G is a topological group".
2. Prove that if G is a topological group then $\pi_1(G)$ is Abelian.

Part B

Problem 4.

1. Let $p : X \to B$ be covering map and let $f : Y \to B$ be a continuous map. State in full the lifting theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a lift of f to a map $\tilde{f} : Y \to X$ such that $f = p \circ \tilde{f}$.
2. Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{it}$. Is it true that every map $f : \mathbb{RP}^2 \to S^1$ can be lifted to a map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}$ such that $f = p \circ \tilde{f}$? Justify your answer.

Problem 5. Let X be a topological space and let A be a subspace of X that has a neighborhood V that deformation retracts back to A. Prove that $\tilde{H}_\star(X/A) \simeq H_\star(X, A)$. Feel free to use excision, the exact sequences associated with pairs and triples of topological spaces and/or any other result proven in class prior to this particular isomorphism.

Problem 6. A Δ-space X is given by $S_2 = \{A, B\} \xrightarrow{\partial_{0,1,2}} S_1 = \{a, b, c\} \xrightarrow{\partial_{0,1}} S_0 = \{P, Q\}$, where $\partial_{0,1,2}(A) = (c, b, a)$, $\partial_{0,1,2}(B) = (c, a, b)$, $\partial_{0,1}(a) = (Q, P)$, $\partial_{0,1}(b) = (Q, P)$ and $\partial_{0,1}(c) = (Q, Q)$.

1. Write down the complex $C_\star^\Delta(X)$ (including the boundary maps).
2. Calculate the homology of X. (I.e., calculate $H_k^\Delta(X)$ for all k).
3. Can you identify the topological space $|X|$?

Problem 7. Let γ be an embedding of the "figure 8 space" $S^1 \vee S^1$ into $\mathbb{R}^3$. Prove that $H_1(\mathbb{R}^3 - \gamma(S^1 \vee S^1)) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Problem 8. Let $f : S^n \to S^n$ be a continous map and let $y \in S^n$ be so that $f^{-1}(y)$ is finite.

1. Define "the degree $\deg(f)$ of f".
2. For $x \in f^{-1}(y)$, define "the local degree $\deg_x(f)$ of f at x".
3. What is the "local degree formula"?
4. If it is also given that f is even (i.e., $f(x) = f(-x)$ for all $x \in S^n$), show that $\deg(f)$ is also even. Be careful to separate the cases where n is even and where n is odd.

Problem 9. Let A, B and C be chain complexes, let f_1 and f_2 be chain-complex morphisms from A to B and let g_1 and g_2 be chain-complex morphisms from B to C.

1. Define "f_1 is homotopic to f_2".
2. Prove that if f_1 is homotopic to f_2 and if g_1 is homotopic to g_2, then $g_1 \circ f_1$ is homotopic to $g_2 \circ f_2$.

Good Luck!

— please stay for the post mortem class discussion —