DEPARTMENT OF MATHEMATICS
University of Toronto

Practice exam in Topology (3 hours)

1. Let X be a compact metric space and let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X. Show that there exists $\varepsilon > 0$ such that for every $x \in X$ there exists $\alpha \in A$ such that the ε-ball centred at X is contained in U_α. (ε is called a *Lebesgue number* for the covering.)

2. Let X be the *topologist's sine curve* defined by

$$X \;=\; \begin{array}{c} \{(x, \sin \pi/x) \mid 0 < x \leq 1\} \cup \{(0,y) \mid -1 \leq y \leq 2\} \\ \cup\{(x,2) \mid 0 \leq x \leq 1\} \cup \{(1,y),\, 0 \leq y \leq 2\} \end{array} \;\subset\; \mathbb{R}^2$$

(i) Sketch X.

(ii) Let $f : X \to X$ be continuous.

Show that either $f(X) = X$ or else there exists $\delta > 0$ such that $f(X) \cap \{(x,y) \mid 0 < x < \delta,\, -\frac{3}{2} \leq y \leq \frac{3}{2}) = \emptyset$.

3. Let $T = S' \times S'$ denote the torus.

(i) Show that T can be covered by 3 contractible open subsets.

(ii) Show that T cannot be covered by 2 contractible open subsets.

4. (i) Define *topological manifold*.

(ii) Define *covering map*.

(iii) Let $p : Y \to X$ be a covering map where X is a topological manifold. Is Y necessarily a topological manifold? Why or why not?

5. (i) Give an example of spaces $A \subset B \subset X$ such that

$$H_*(X - A, B - A) \ncong H_*(X, B) \ .$$

(ii) State conditions on the subspaces A, B which guarantee that

$$H_*(X - A, B - A) \ \cong \ H_*(X, B) \ .$$

6. Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of chain complexes. Define the connecting homomorphism $\Delta : H_n(C) \to H_{n-1}(A)$. Show that the sequence $H_n(B) \xrightarrow{j} H_n(C) \xrightarrow{\Delta} H_{n-1}(A)$ is exact at $H_n(C)$.