DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

January 1995

1. Let X be connected and let C be a closed subset such that the boundary of C is a single point. Show that C is connected.

2. Let X be the *comb space* defined by

$$X = \Big(\bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) \mid 0 \le y \le 1\}\Big) \cup \{(0,y) \mid 0 \le y \le 1\} \cup \{(x,0) \mid 0 \le x \le 1\} \subset \mathbb{R}^2 .$$

Let $I = \{(0,y) \mid 0 \le y \le 1\} \subset X$.

(i) Sketch X.

(ii) Define *deformation retract* and *strong deformation retract.*

(iii) Show that I is a deformation retract of X.

(iv) Show that I is not a strong deformation retract of X.

3. (i) Suppose $n \ge 2$. Does there exist a continuous map $f : S^n \to S^1$ which is not homotopic to a constant?

(ii) Suppose $n \ge 2$. Does there exist a continuous map $f : \mathbb{R}P^n \to S^1$ which is not homotopic to a constant?

(iii) Let $T = S^1 \times S^1$ be the torus. Does there exist a continuous map $f : T \to S^1$ which is not homotopic to a constant?

In each case, carefully justify your answer.

4. Let $p : E \to X$ be a covering map and let $f : Y \to X$ be any continuous map. Let $P = \{(y, e) \in Y \times E \mid fy = pe\}$ and define $\pi : P \to Y$ by $\pi(y, e) = y$. Show that π is a covering map.

5. Let $f : S^{n-1} \to Y$ be continuous $(n > 1)$ and let $Y_f = D^n \cup_f Y$. Show that:

(i) $H_m(Y) \cong H_m(Y_f)$ for $m \neq n, n-1$.

(ii) There is an exact sequence

$$0 \to H_n(Y) \to H_n(Y_f) \to H_{n-1}(S^{n-1}) \to H_{n-1}(Y) \to H_{n-1}(Y_f) \to 0 .$$

6. Compute $H_*(S^n)$.