DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

September 1995

1. (a) Define the "quotient topology".

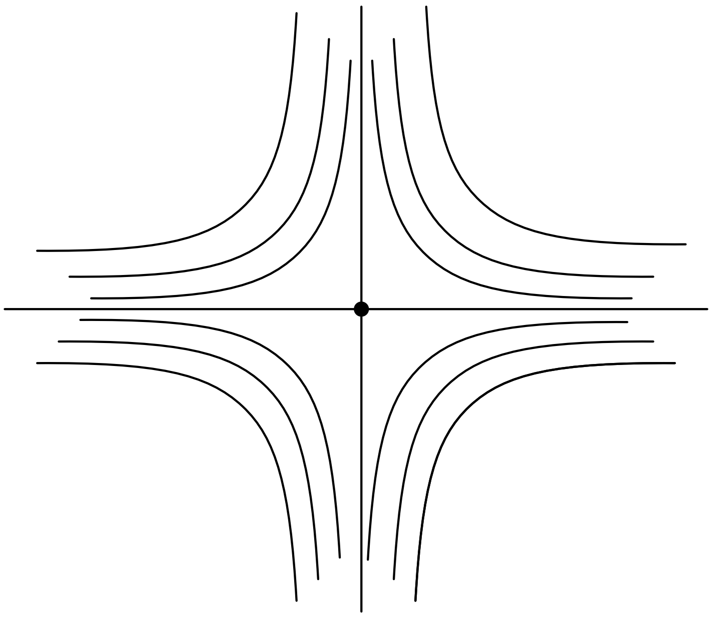
   (b) Consider the system of differential equations:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y, \end{aligned}$$

   $(x, y) \in \mathbb{R}^2$.

   The general solution is $x = ae^t$, $y = be^{-t}$.

   The points on any orbit (solution curve) satisfy $xy = c$ for some c. The orbits are drawn below.



There are 9 possible types of orbits, given explicitly as follows:

for $c > 0$
$$T_1(c) = \{(x,y) \in \mathbb{R}^2 \mid xy = c \text{ and } x > 0,\ y > 0\}$$
$$T_2(c) = \{(x,y) \in \mathbb{R}^2 \mid xy = c \text{ and } x < 0,\ y < 0\}$$

for $c < 0$
$$T_3(c) = \{(x,y) \in \mathbb{R}^2 \mid xy = c \text{ and } x > 0,\ y < 0\}$$
$$T_4(c) = \{(x,y) \in \mathbb{R}^2 \mid xy = c \text{ and } x < 0,\ y > 0\}$$

for $c = 0$
$$T_5 = \{(x,0) \in \mathbb{R}^2 \mid x > 0\}$$
$$T_6 = \{(0,y) \in \mathbb{R}^2 \mid y > 0\}$$
$$T_7 = \{(x,0) \in \mathbb{R}^2 \mid x < 0\}$$
$$T_8 = \{(0,y) \in \mathbb{R}^2 \mid y < 0\}$$
$$T_9 = \{(0,0)\}.$$

The orbit space $\mathcal{M}$ is defined as $\mathcal{M} = \frac{\mathbb{R}^2}{\sim}$ with the quotient topology where $(x,y) \sim (x',y')$ if they lie on the same orbit.

Describe $\mathcal{M}$ as a topological space; that is, give a basis for the open sets. Is $\mathcal{M}$ Hausdorff? Why or why not?

2. Let X be a path connected space. Suppose that there exists a continuous map m: $X \times X \to X$ and a point $e \in X$ such that $m(e,x) = m(x,e) = x$ for all $x \in X$. Show that the fundamental group $\pi_1(X,e)$ is abelian.

3. (a) Define what it means for a topological space to be "normal".

   (b) State Urysohn's Lemma.

   (c) Prove that a compact Hausdorff space is normal.

4. Let sq: $S^1 \to S^1$ be the map $z \mapsto z^2$ where the circle is regarded as the unit ball of $\mathbb{C}$ and the multiplication is that from $\mathbb{C}$. Compute the group homomorphism $sq_\sharp$: $\pi_1(S^1) \to \pi_1(S^1)$.

5. Let M be an n-dimensional path connected topological manifold (so that each point has a neighbourhood which is homeomorphic to $\mathbb{R}^n$), and let $x \in M$. Compute the relative homology $H_q(M, M-x)$ for all q.

6. Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of chain complexes. Define the connecting homomorphism $\Delta : H_n(C) \to H_{n-1}(A)$. Show that the sequence $H_n(B) \xrightarrow{j} H_n(C) \xrightarrow{\Delta} H_{n-1}(A)$ is exact at $H_n(C)$.