DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

January 1997

No aids.
Do all questions.
Each question is worth 20 marks.

1. a) Let $f\colon S^2 \times I \to \mathbb{R}$ be continuous. Suppose that $\min\limits_{p\in S^2} f(p,0) = 0$, $\min\limits_{p\in S^2} f(p,1) = 1$. Prove that for every $0 < m < 1$, $\exists\ 0 < t < 1$ such that $\min\limits_{p\in S^2} f(p,t) = m$.

   b) Does the result in part (a) hold if S^2 is replaced by $\mathbb{R}$? Explain your answer.

2. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of Hausdorff spaces where X_i is a closed subspace of X_{i+1} for each i. Let $X = \bigcup\limits_{i=1}^{\infty} X_i$. Define the coherent topology on X by $U \subset X$ is open $\Leftrightarrow U \cap X_i$ is open in $X_i\ \forall i$.

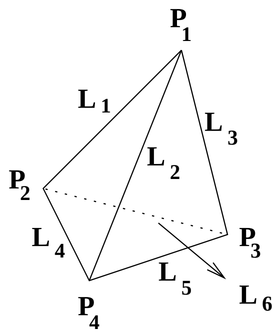
   a) Verify that this is a topology.

   b) Show that X_i is a subspace of X in this topology.

   c) Suppose that each X_i is normal; state Tietze's extension theorem and use it to show that X is normal.

3. Let M be a compact connected manifold. (Recall that a manifold is Hausdorff and locally Euclidean, that is, every point has a neighbourhood homeomorphic to an open set in $\mathbb{R}^n$ for some n.) Let $\pi\colon P \to M$ be the universal cover. Show that P is compact $\Leftrightarrow \pi_1(M)$ is finite.

4. Let X be the outline of the tetrahedron; that is, $X = \bigcup_{i=1}^{6} L_i \bigcup \bigcup_{i=1}^{4} P_i$



where L_i are the edges and P_i are the vertices. Calculate $H_*(X;\mathbb{Z})$.

5. a) Calculate $\pi_1(S^2 \times S^1)$.

   b) Calculate $\pi_1(S^2 \times \Delta)$ where $\Delta = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

   c) Notice that $S^2 \times S^1 \underset{\text{homeo}}{\cong} S^2 \times \partial\Delta$. Show that $S^2 \times S^1$ is not a retract of $S^2 \times \Delta$.