DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

September 1998

(1) Let X be the "comb" space defined by

$$X = \Big(\bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) \mid 0 \le y \le 1\}\Big) \cup \{(0,y) \mid 0 \le y \le 1\} \cup \{(x,0) \mid 0 \le x \le 1\} \subset \mathbb{R}^2 .$$

Let

$$I = \{(0,y) \mid 0 \le y \le 1\} \subset X .$$

(a) Prove that I is a deformation retract of X.

(b) A subspace A of a topological space Y is a *strong deformation retract* if there exists a continuous map

$$r : Y \to A$$

such that $ri = 1_A$ and $ir \simeq 1_Y$ rel A, where $i : A \to Y$ is the inclusion map. Show that I is not a strong deformation retract of X.

(2) Let X and Y be topological spaces. Let $A \subset X$ be closed and let $f : A \to Y$ be continuous. Define

$$Z_f := (X \amalg Y)/\sim$$

where $a \sim f(a)$ for all $a \in A$.

(a) Show that if X and Y are Hausdorff then Z_f is Hausdorff also.

(b) Define what it means for a topological space to be normal.

(c) State Urysohn's lemma.

(d) Show that if X and Y are normal then Z_f is normal also.

(3) (a) Prove that for $n > 1$, $O(n)$ and $GL(n, \mathbb{R})$ are homotopy equivalent.

(b) Show that $O(n)$ has precisely 2 path components.

(4) Let $T^2 := S^1 \times S^1$ be the 2-torus. Let

$$U = (S^1 \times \{1\}) \cup (\{1\} \times S^1) \subset T^2$$

and let $i : U \to T^2$ denote the inclusion map.

(a) Compute $\pi_1(U)$, $\pi_1(T^2)$ and $i_* : \pi_1(U) \to \pi_1(T^2)$.

(b) What is the kernel of i_*?

(5) Give a CW structure (i.e. cell decomposition) of $\mathbb{C}P^n$ with the minimum possible number of cells. How do you know that your decomposition is minimal?

(6) (a) Let $f, g : S^n \to S^n$ be continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Show that $f \simeq a \circ g$, where a is the antipodal map.

(b) Prove that any continuous map $f : S^{2n} \to S^{2n}$ either has a fixed point or there is a point x with $f(x) = -x$.

(c) Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.