DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

September 13, 1999

No aids.
Do all questions.
Questions are of equal value.
You may take the homology/cohomology of spheres and projective spaces (both real and complex) as "known" and quote them without proof wherever needed.

1. Let $X_1 \subset X_2 \subset \ldots$ be a sequence of Hausdorff spaces where X_i is a closed subspace of X_{i+1} for each i. Let $X = \cup_{i=1}^{\infty} X_i$. Define the coherent topology on X by declaring that a subset A of X is closed whenever $A \cap X_i$ is closed for all i.

(a) Verify that this is a topology.

(b) Define what it means for a topological space to be normal.

(c) State Urysohn's Lemma.

(d) State the Tietze Extension Theorem and use it to show that if X_i is normal for all i then X is normal.

2. There are two main methods of calculating fundamental groups: using covering spaces or using Van Kampen's Theorem. Describe each method (including appropriate definitions and statements of relevant theorems) and give an example of the use of each method.

3. (a) Let X be a topological space whose fundamental group is finite. Use the fact that any group homomorphism from a finite group to $\mathbf{Z}$ is trivial to show that any continuous function from X to S^1 is null homotopic. Give precise statements of any theorems you use.

(b) If S^1 were replaced in part (a) by some other space whose fundamental group is $\mathbf{Z}$, would the same conclusion be valid?

4. Let M be an n-dimensional manifold and let x be a point in M. Calculate $H_q(M, M - x)$ for all q.

5. (a) Define complex projective space $\mathbf{C}P^n$.

   (b) What is the cohomology of $\mathbf{C}P^n$?

   Note: As per instructions above, you need only state the answer; proof is not required.

   (c) Give an example of a space which has the same cohomology groups as $\mathbf{C}P^2$ but which is not homotopy equivalent to $\mathbf{C}P^2$. How do you know that your space is not homotopy equivalent to $\mathbf{C}P^2$?

6. (a) What does it mean to say that a manifold is orientable?

   Note: There is more than one possible answer to this question. Some formulations require the existence of a smooth structure on the manifold; you may assume it is a differentiable manifold if you wish.

   (b) Show that real projective space $\mathbf{R}P^n$ is orientable if and only if n is odd.